Approximation algorithms for two-machine proportionate routing open shop on a tree

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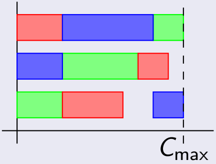
Open shop problem

Open Shop($Om||C_{max}$)

Machines: $M_1, M_2 \ldots M_m$

Jobs: $J_1, J_2 \ldots J_n$



- $O2||C_{max}$ is polynomially solvable (Gonzalez, Sahni 1976)
- $O3||C_{max}$ is NP-hard (Gonzalez, Sahni 1976)

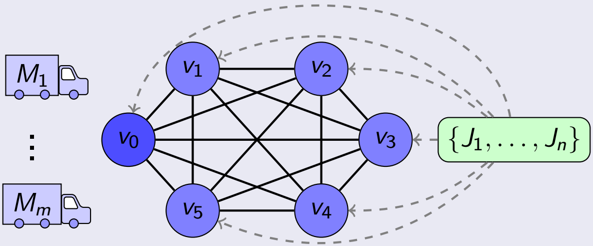
Proportionate Open Shop problem

	J_1	J_2	J_3	$\dots$	J_n
M_1	p_{11}	p_{12}	p_{13}	$\dots$	p_{1n}
M_2	p_{21}	p_{22}	p_{23}	$\dots$	p_{2n}
M_3	p_{31}	p_{32}	p_{33}	$\dots$	p_{3n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$
M_m	p_{m1}	p_{m2}	p_{m3}	$\dots$	p_{mn}

Proportionate Open Shop ($Om|j\text{−prpt}|C_{\max}$): $p_{ij} = p_j$

- $O3|j\text{−prpt}|C_{\max}$ is NP-hard (Lui, Bulfin 1987)
- $O3|j\text{−prpt}|C_{\max}$ can be solved in pseudopolynomial time (Sevastyanov 2019)

Routing Open shop model



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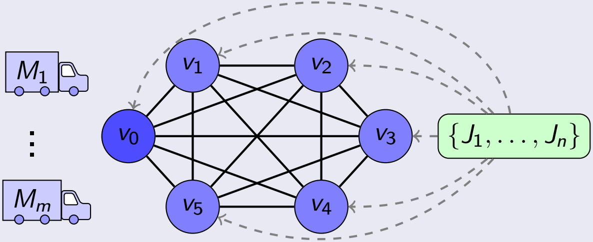

$ROm|\text{j}-\text{prpt}, G = X|R_{\max}$

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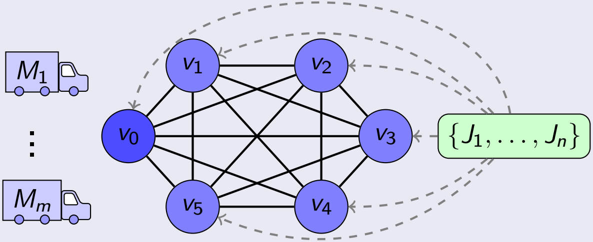

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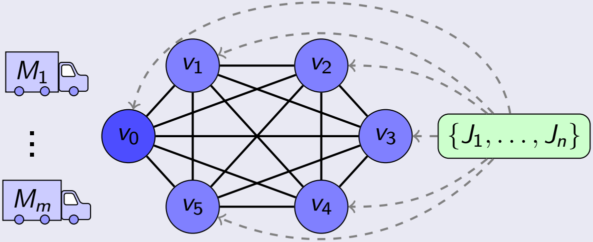

$$ROm|\text{j}{-}\text{prpt}, G = X|R_{\max}$$

$$ROm|\text{j}{-}\text{prpt}, Rtt, G = X|R_{\max}$$

$$\overrightarrow{R}Om|\text{j}{-}\text{prpt}, G = X|R_{\max}$$

Standard lower bound

	J_1	J_2	J_3	$\dots$	J_n	
M_1	p_{11}	p_{12}	p_{13}	$\dots$	p_{1n}	l_1
M_2	p_{21}	p_{22}	p_{23}	$\dots$	p_{2n}	l_2
M_3	p_{31}	p_{32}	p_{33}	$\dots$	p_{3n}	l_3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$
M_m	p_{m1}	p_{m2}	p_{m3}	$\dots$	p_{mn}	l_m
	d_1	d_2	d_3	$\dots$	d_n	

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Lower bound for $Om||C_{\max}$

$$\bar{C} = \max\{\ell_{\max}, d_{\max}\}$$

Standard lower bound

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Lower bound for $Om||C_{\max}$

$$\bar{C} = \max\{\ell_{\max}, d_{\max}\}$$

Lower bound for $ROm||R_{\max}$

$$\bar{R} = \max\{\ell_{\max} + T^*, \max_{v \in V}(d_{\max}(v) + 2dist(v_0, v))\}$$

where $dist(v_i, v_j)$ is the distance between nodes v_i and v_j.

Standard lower bound

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Lower bound for $RO2|Rtt|R_{\max}$

$$\bar{R} = \max\Big\{\max_i(\ell_i + T_i^*),$$

$$\max_{v \in V}\left(d_{\max}(v) + dist_1(v_0, v) + dist_2(v_0, v)\right)\Big\}$$

Standard lower bound

Lower bound for $RO2|Rtt|R_{\max}$

$$\bar{R} = \max\left\{\max_i(\ell_i + T_i^*),\right.$$

$$\left.\max_{v \in V}\left(d_{\max}(v) + dist_1(v_0, v) + dist_2(v_0, v)\right)\right\}$$

Lower bound for $\overrightarrow{R}O2|Rtt|R_{\max}$

$$\bar{R} = \max\left\{\max(\ell_i + T_i^*), \max_{v \in V}(d_{\max}(v) + \overleftrightarrow{dist}_{\min}(v_0, v))\right\},$$

$$\overleftrightarrow{dist}_{\min}(v, u) = \min\left\{dist_1(v, u) + dist_2(u, v), dist_2(v, u) + dist_1(u, v)\right\}$$

Optima Localization Problem

Optima Localization Problem

Find minimal value ρ such that for all instances I: $R^*_{\max}(I) \in [\bar{R}, \rho\bar{R}]$.

Optima localization intervals for $RO2||R_{\max}$:

- $[\bar{R}, \frac{6}{5}\bar{R}]$ for $RO2|G = K_2|R_{\max}$ problem (Averbakh et al 2005)
- $[\bar{R}, \frac{6}{5}\bar{R}]$ for $RO2|G = K_3|R_{\max}$ problem (Chernykh, Lgotina 2016)
- $[\bar{R}, \frac{6}{5}\bar{R}]$ for $\overrightarrow{R}O2|G = tree|R_{\max}$ problem (Chernykh, Krivonogova 2019)

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Optima localization intervals for $RO2|Rtt|R_{\max}$:

- $[\bar{R}, \frac{5}{4}\bar{R}]$ for $RO2|Rtt, G = K_2|R_{\max}$ и $RO2|Rtt, G = K_3|R_{\max}$ problem (Chernykh, Lgotina 2019)
- $[\bar{R}, \frac{5}{4}\bar{R}]$ for $\overrightarrow{R}O2|Rtt, G = tree|R_{\max}$ problem (Chernykh, Krivonogova 2019)

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Find minimal value ρ such that for all instances I: $R^*_{\max}(I) \in [\bar{R}, \rho\bar{R}]$.

Optima localization intervals for proportionate problems:

- $[\bar{C}, \frac{10}{9}\bar{C}]$ for $O3|j-prpt|C_{\max}$ problem (Sevastyanov 2019)
- $[\bar{R}, \frac{7}{6}\bar{R}]$ for $RO2|j-prpt, G=K_3|R_{\max}$ and $RO2|j-prpt, G=K_2|R_{\max}$ problems (Pyatkin, Chernykh, 2022)

Optima Localization Problem

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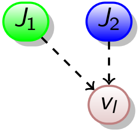
Problem	Opt. loc.	Problem with Qtt/Rtt	Opt. loc.
$RO2 \mid G = K_2 \mid R_{\max}$	$[\bar{R}, 6/5\bar{R}]$	$RO2 \mid G = K_2, Rtt \mid R_{\max}$	$[\bar{R}, 5/4\bar{R}]$
$RO2 \mid G = K_3 \mid R_{\max}$	$[\bar{R}, 6/5\bar{R}]$	$RO2 \mid G = K_3, Rtt \mid R_{\max}$	$[\bar{R}, 5/4\bar{R}]$
$RO2 \mid G = tree \mid R_{\max}$	$[\bar{R}, 6/5\bar{R}]$	$RO2 \mid G = tree, Rtt \mid R_{\max}$	$[\bar{R}, 5/4\bar{R}]$
$RO2 \mid j\text{-}prpt, G = K_2 \mid R_{\max}$	$[\bar{R}, 7/6\bar{R}]$	$RO2 \mid j\text{-}prpt, G = K_2, Rtt \mid R_{\max}$	?
$RO2 \mid j\text{-}prpt, G = K_3 \mid R_{\max}$	$[\bar{R}, 7/6\bar{R}]$	$RO2 \mid j\text{-}prpt, G = K_3, Rtt \mid R_{\max}$	?

Job aggregation



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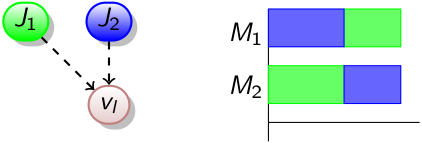

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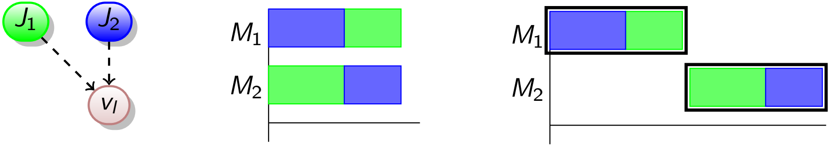

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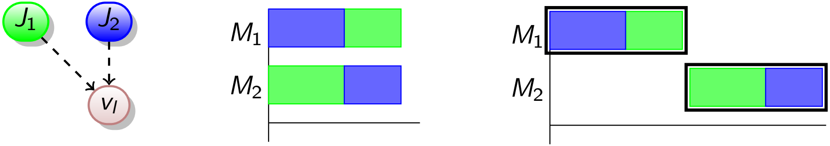
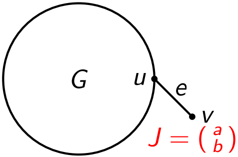


Definition

A node v is overloaded, if

$$\Delta(v) > \bar{R}(I) - \overleftrightarrow{dist}_{\min}(v_0, v)$$

Terminal edge contraction



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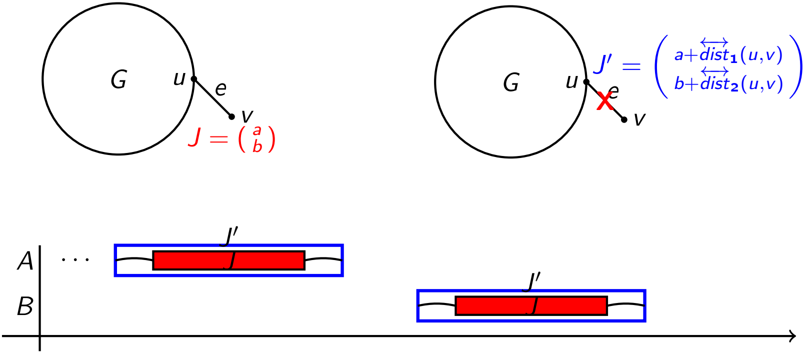
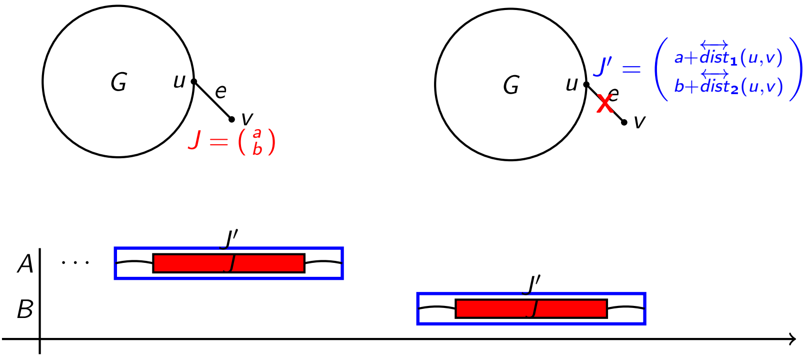


Terminal edge contraction



Definition

Terminal edge e is overloaded, if

$$a + b + \overleftrightarrow{dist}_1(u, v) + \overleftrightarrow{dist}_2(u, v) + \overleftrightarrow{dist}_{\min}(v_0, u) > \bar{R},$$

Instance reduction procedure

Algorithm $\mathcal{A}$:

1. **For each** underloaded $v \in V$ perform the job aggregation of the set $\mathcal{J}(v)$.
2. **For each** terminal node $v \neq v_0$ with single job and its incident edge $e = [u, v]$
   1. **If** e is underloaded, then
      1. Perform the contraction of e,
      2. **If** u is underloaded,, **then** perform the job aggregation of the set $\mathcal{J}(u)$.
3. **If** v is overloaded **then** perform the job aggregations in $\mathcal{J}(v)$ to obtain an irreducible instance.

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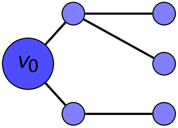
3. **If** v is overloaded **then** perform the job aggregations in $\mathcal{J}(v)$ to obtain an irreducible instance.

Lemma

Any instance of $\overrightarrow{R}O2|Rtt|R_{\max}$ contains at most one overloaded element (node or edge).

Possible outputs of algorithm $\mathcal{A}$ for $G = tree$

Case 1:



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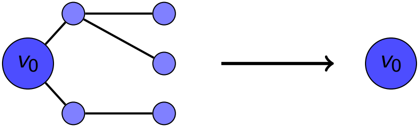

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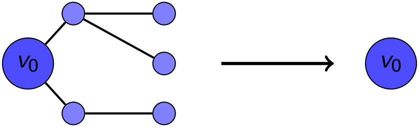
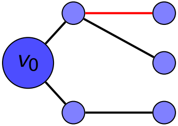


Case 2:



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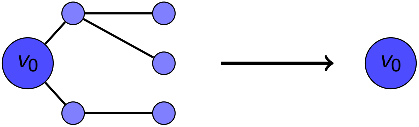

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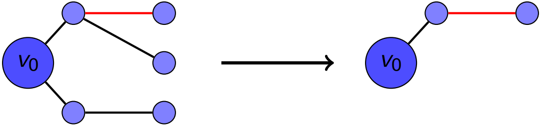

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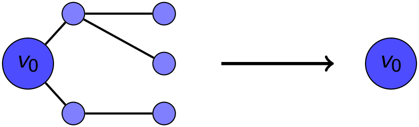

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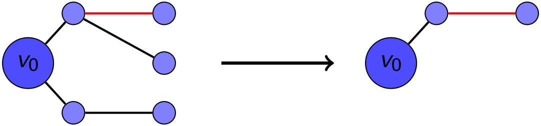
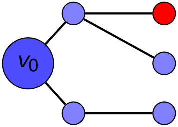


Case 3:



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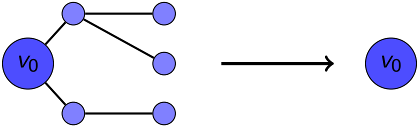

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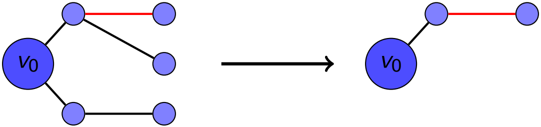

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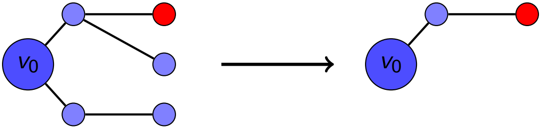

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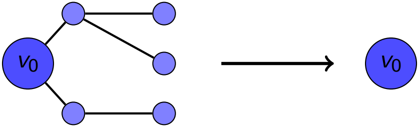

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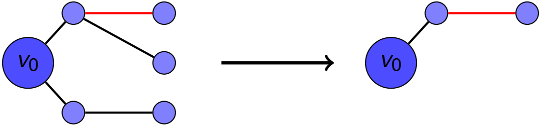

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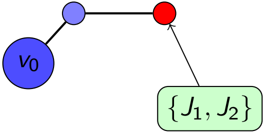

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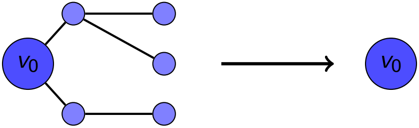

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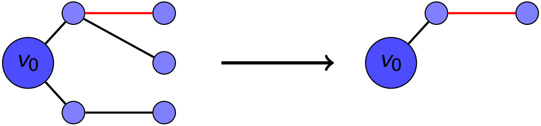

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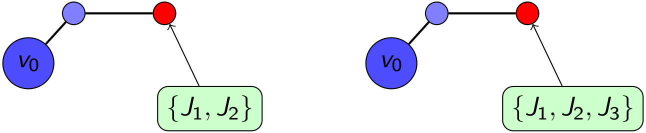

Previous results

Theorem (Chernykh I., Lgotina E., 2021)

Let I be an instance of the $RO2|G = tree|R_{\max}$ problem, $\tilde{I}$ is obtained from I by the algorithm $\mathcal{A}$, and one of the following conditions is true:

1. $G(\tilde{I})$ has a single node v_0,
2. $G(\tilde{I})$ is a chain, connecting v_0 with an overloaded node v, which contains exactly three jobs,
3. $G(\tilde{I})$ is a chain, connecting v_0 with a node v with single job at each node, and the edge incident to v is overloaded.

Then one can in linear time build a feasible schedule $S(\tilde{I})$ such that $R_{\max}(S) = \bar{R}(I)$.

Main results: identical travel times

Theorem 1

For any instance of the $\overrightarrow{R}O2|j\text{-}prpt, G = tree|R_{\max}$ problem a feasible schedule S with $R_{\max}(S) \leqslant \frac{7}{6}\bar{R}$ can be constructed in linear time $O(n)$.

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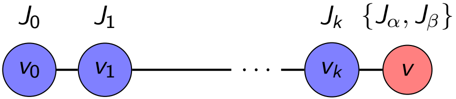

Main results: unrelated travel times

Terminal edge contraction operation would violate the *j-prpt* property.

$$J' = \begin{pmatrix} a + \overleftrightarrow{dist}_{\mathbf{1}}(u,v) \\ b + \overleftrightarrow{dist}_{\mathbf{2}}(u,v) \end{pmatrix}$$

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Replace $dist_1(u,v)$ and $dist_2(u,v)$ with the $\max\{dist_1(u,v), dist_2(u,v)\}$

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$$\bar{R}(I) = \max\left\{\ell + T_1^*, \max_{v \in V}(d_{\max}(v) + \overleftrightarrow{dist}_{\min}(v_0, v))\right\}$$

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Definition

Distance functions $dist_1$ and $dist_2$ are called *comparable*, if

$$\forall u, v \in V \;\; dist_1(u, v) \geqslant dist_2(u, v).$$

Main results: unrelated travel times

Theorem 2

For any instance I of the $\overrightarrow{R}O2|j\text{-}prpt, Rtt, G = tree|R_{\max}$ with comparable distances a feasible schedule S with $R_{\max}(S) \leqslant \frac{7}{6}\bar{R}$ can be built in linear time.

Main results and future research

Problem	Opt. loc.	Problem with Qtt/Rtt	Opt. loc.
$RO2 \vert G = K_2 \vert R_{\max}$	$[\bar{R}, 6/5\bar{R}]$	$RO2 \vert G = K_2, Rtt \vert R_{\max}$	$[\bar{R}, 5/4\bar{R}]$
$RO2 \vert G = K_3 \vert R_{\max}$	$[\bar{R}, 6/5\bar{R}]$	$RO2 \vert G = K_3, Rtt \vert R_{\max}$	$[\bar{R}, 5/4\bar{R}]$
$RO2 \vert G = tree \vert R_{\max}$	$[\bar{R}, 6/5\bar{R}]$	$RO2 \vert G = tree, Rtt \vert R_{\max}$	$[\bar{R}, 5/4\bar{R}]$
$RO2 \vert j\text{-}prpt, G = K_2 \vert R_{\max}$	$[\bar{R}, 7/6\bar{R}]$	$RO2 \vert j\text{-}prpt, G = K_2, Rtt \vert R_{\max}$	$[\bar{R}, 7/6\bar{R}]$
$RO2 \vert j\text{-}prpt, G = K_3 \vert R_{\max}$	$[\bar{R}, 7/6\bar{R}]$	$RO2 \vert j\text{-}prpt, G = K_3, Rtt \vert R_{\max}$	?
$RO2 \vert j\text{-}prpt, G = tree \vert R_{\max}$	$[\bar{R}, 7/6\bar{R}]$	$RO2 \vert j\text{-}prpt, G = tree, Rtt \vert R_{\max}$	?

Thank you for attention!