The golden ratio and normality in two-machine routing open shop

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Open shop problem settings

$Om||C_{\max}$:

- Machines $\{M_1, \ldots, M_m\}$;
- Jobs $\{J_1, \ldots, J_n\}$;
- Each job has to be processed by each machine in arbitrary order;
- Processing times $P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ \vdots & & & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mn} \end{pmatrix}$;
- Objective function $C_{\max}$: the makespan.

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Short review

- $O2||C_{\max}$ is solvable in $O(n)$ [Gonzalez, Sahni 1976];
- $O3||C_{\max}$ is $\mathbb{NP}$-hard (but is it strongly $\mathbb{NP}$-hard?) [Gonzalez, Sahni 1976];
- $O||C_{\max}$ is strongly $\mathbb{NP}$-hard, can't be approximated better that $\frac{5}{4}$ [Williamson *et al* 1997].

A special case

Proportionate setting

Restriction on processing times: $p_{ij} = p_j$, $P = \begin{pmatrix} p_1 & p_2 & \cdots & p_n \\ \vdots & & & \vdots \\ p_1 & p_2 & \cdots & p_n \end{pmatrix}$

Commonly used notation: *prpt* (*e.g.*, $Om|prpt|C_{\max}$).

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m-prpt

- $O3|j\text{-}prpt|C_{\max}$ is $\mathbb{NP}$-hard [Lui, Bulfin 1987];
- $O3|j\text{-}prpt|C_{\max}$ is pseudopolynomially solvable [Sevastyanov 2019].

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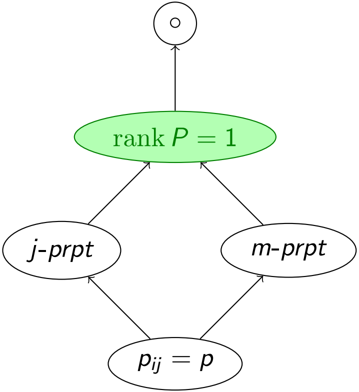

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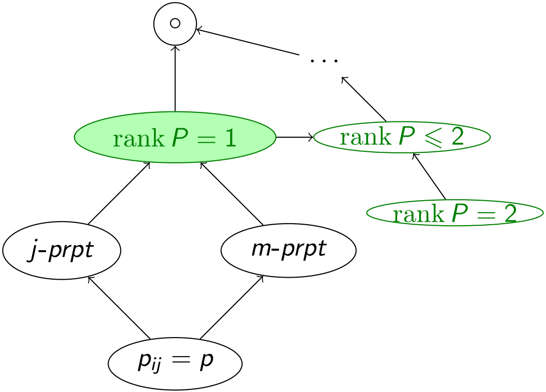

Standard lower bound

	J_1	J_2	$\dots$	J_n	
M_1	p_{11}	p_{21}	$\dots$	p_{n1}	
$\vdots$				$\vdots$	
M_m	p_{1m}	p_{2m}	$\dots$	p_{nm}	

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	J_1	J_2	$\dots$	J_n	Σ
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Lower bound on the makespan

$$C^*_{\max} \geqslant \bar{C} \doteq \max_{i,j}\{\ell_i, d_j\} = \max\{\ell_{\max}, d_{\max}\}.$$

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Definition

- A feasible schedule S is normal, if $C_{\max}(S) = \bar{C}$. Instances admitting construction of a normal schedule a also called normal.
- Abnormality of instance I is $\alpha(I) = \dfrac{C^*_{\max}(I)}{\bar{C}(I)}$.
- If $\mathcal{K}$ is a set of instances,

$$\alpha(\mathcal{K}) = \sup_{I \in \mathcal{K}} \alpha(I).$$

Review

Notation

- $\mathcal{I}_m$ is the class of non-trivial instancesfor $Om||C_{\max}$.
- $\mathcal{I}_m(\mathbb{P}) = \{I \in \mathcal{I}_m|\mathbb{P}\}$, where $\mathbb{P}$ — some property/properties.

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- $\mathcal{I}_m$ is the class of non-trivial instancesfor $Om||C_{\max}$.
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Examples of normal classes of instances

- $\mathcal{I}_2$ [Gonzalez, Sahni 1976]
- $\mathcal{I}_3(\ell_{\max} \geqslant 7p_{\max})$ [Sevastyanov 1996]
- $\mathcal{I}_m(\ell_1 \geqslant \max\limits_{i=2,\ldots,m} \ell_i + mp_{\max})$ [Sevastyanov, Ch 1996]
- $\mathcal{I}_3(\ell_1 \geqslant \ell_2 \geqslant \ell_3 + 2p_{\max})$ [Kononov *et al* 1999]
- $\mathcal{I}_3(\ell_{\max} \geqslant 3p_{\max}, \nu = 2)$ [Kashirskikh *et al* 2001]
- $\mathcal{I}_m(j\text{-}prpt, \ell_{\max} \leqslant (m-1)p_{\max})$ [Sevastyanov 2019]
- $\mathcal{I}_3(j\text{-}prpt, \ell_{\max} \leqslant 2.5p_{\max})$ [Sevastyanov 2019]
- ...

Abnormality

- $\alpha(\mathcal{I}_2) = 1$ [Gonzalez, Sahni 1976]
- $\alpha(\mathcal{I}_3) = \frac{4}{3}$ [Sevastyanov, Ch 1998]
- $\alpha(\mathcal{I}_3(\nu = 2)) = \frac{5}{4}$ [Lisitsyna, 2008]
- $\alpha(\mathcal{I}_3(j\text{-}prpt)) = \frac{10}{9}$ [Sevastyanov 2019]
- $\alpha(\mathcal{I}_3(\text{superoverloaded})) = \frac{7}{6}$ [Ch, Pyatkin 2021]
- $\alpha(\mathcal{I}_m(\Delta \leqslant 2\bar{C})) = 1$ [Sevastyanov, Ch]
- $\alpha(\mathcal{I}_m) < 2$ [dense schedules]
- $\dots$

The routing open shop problem

Open Shop ($Om||C_{\max}$)...

Machines	M_1	$\cdots$	M_m
Jobs	J_1	$\cdots$	J_n

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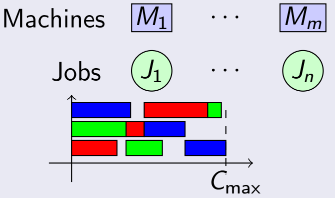

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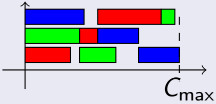
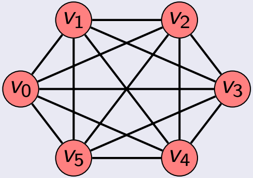


and the metric TSP...

$G = \langle V; E \rangle$



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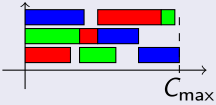

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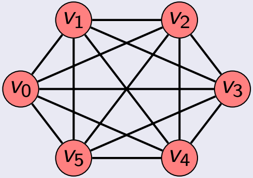

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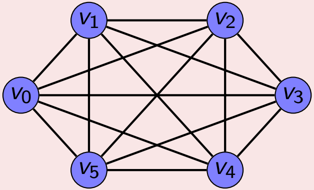

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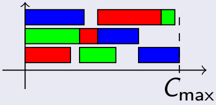

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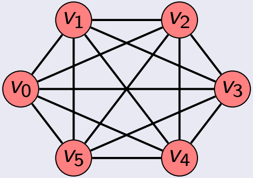

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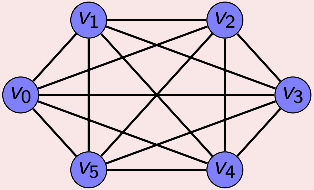

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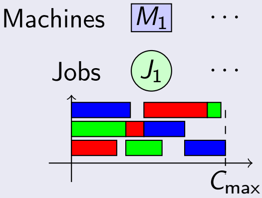

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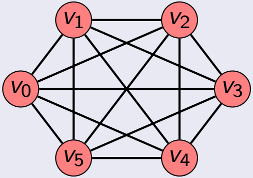

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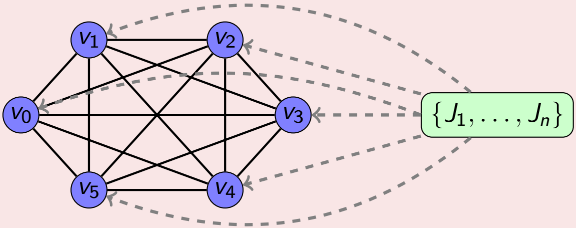

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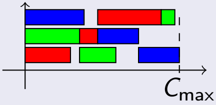

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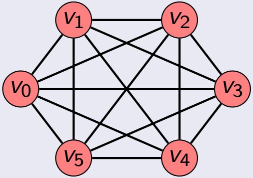

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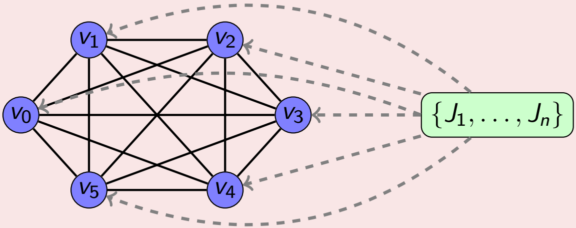
$\vdots$

M_m



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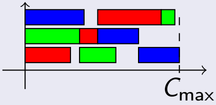

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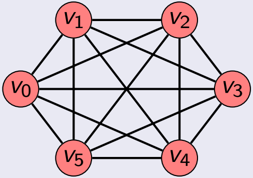

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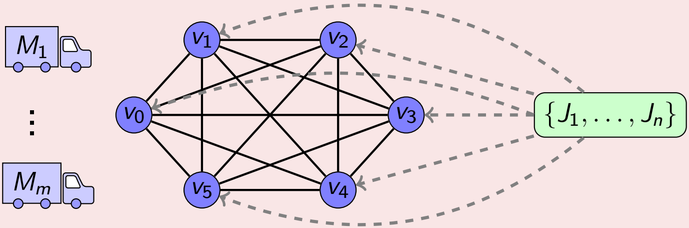

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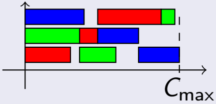

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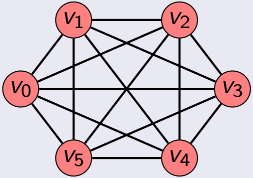

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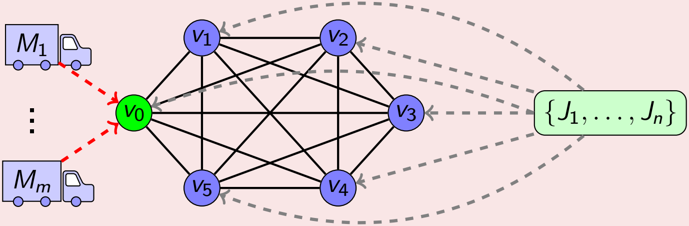

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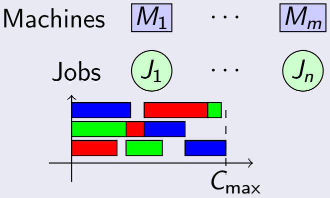

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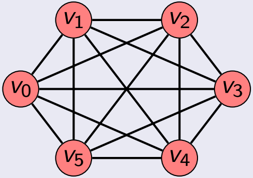

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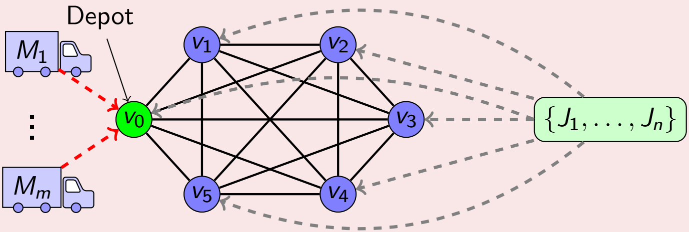

Routing open shop

Notation

- The problem: $ROm||R_{\max}$ or $ROm|G = X|R_{\max}$
- Class of instances of $ROm||R_{\max}$: $\mathcal{I}_m^R$

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- $\mathrm{dist}(u, v)$ — machine travel time between u and v,
- $d_{\max}(v) = \max\limits_{J_j \in \mathcal{J}(v)} d_j$ — maximum job duration in v,
- T^* — optimal route weight G.

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Standard lower bound for $ROm||R_{\max}$

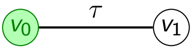
$$\bar{R} = \max\left\{ \ell_{\max} + T^*, \max_{v \in V}\left(d_{\max}(v) + 2\mathrm{dist}(v_0, v)\right)\right\}$$

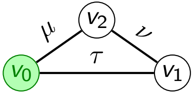
Review

- $RO1||R_{\max}$ is equivalent to the metric TSP;
- $ROm|G = K_1|R_{\max}$ is equivalent to $Om||C_{\max}$;
- $RO2|G = K_2|R_{\max}$ is $\mathbb{NP}$-hard [Averbakh *et al* 2006]
- For $RO2|G = K_2|R_{\max}$ FPTAS exists [Kononov 2012]
- $RO2|j\text{-}prpt, G = K_2|R_{\max}$ is $\mathbb{NP}$-hard [Pyatkin, Ch 2022]
- $\alpha(\mathcal{I}_2^R(G = K_2)) = \frac{6}{5}$ [Averbakh *et al* 2005]
- $\alpha(\mathcal{I}_2^R(G = K_3)) = \frac{6}{5}$ [Ch, Lgotina 2016]
- $\alpha(\mathcal{I}_2^R(G = tree)) = \frac{6}{5}$ [Ch, Krivonogova 2019]
- $\alpha(\mathcal{I}_2^R(j\text{-}prpt, G = K_3)) = \frac{7}{6}$ [Pyatkin, Ch 2022]
- $\alpha(\mathcal{I}_2^R(j\text{-}prpt, G = tree)) = \frac{7}{6}$ [Shmyrina 2022]

The object of investigation

$RO2|\mathrm{rank}\, P = 1, G = K_2|R_{\max}.$ $RO2|\mathrm{rank}\, P = 1, G = K_3|R_{\max}.$





Processing times: $P = \begin{pmatrix} a_1 & \dots & a_n \\ b_1 & \dots & b_n \end{pmatrix} = \begin{pmatrix} kp_1 & \dots & kp_n \\ p_1 & \dots & p_n \end{pmatrix}$, without loss of generality $k \geqslant 1$.

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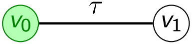

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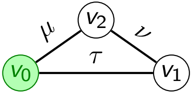

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Questions:

1. Which values of k guarantee normality?
2. Abnormality $\alpha(\mathcal{I}_2^R(\mathrm{rank}\, P = 1, G = K_2))$ as function of k.
3. Abnormality $\alpha(\mathcal{I}_2^R(\mathrm{rank}\, P = 1, G = K_3))$ as function of k.

Lemma 1

Let $I \in \mathcal{I}_2^R(\mathrm{rank}\, P = 1, G = K_2)$ is an instance with $k \geqslant \Phi$, where Φ is the golden ratio. When I is normal.

The boundary on normality

Lemma 1

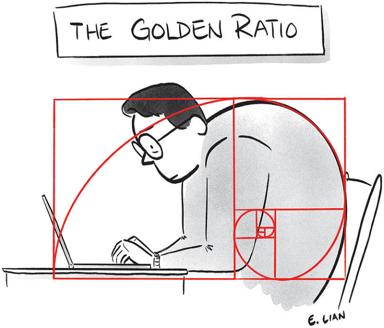
Let $I \in \mathcal{I}_2^R(\mathrm{rank}\, P = 1, G = K_2)$ is an instance with $k \geqslant \Phi$, where Φ is the golden ratio. When I is normal.

Φ — positive root of $x^2 - x = 1$;

$$\Phi = \frac{\sqrt{5}+1}{2} \approx 1,618...$$

$$k \geqslant \Phi \Rightarrow k^2 - k \geqslant 1,$$

$$k \in [1, \Phi) \Rightarrow k^2 - k < 1.$$



The bounary is tight

Lemma 1

Let $I \in \mathcal{I}_2^R(\mathrm{rank}\, P = 1, G = K_2)$ is an instance with $k \geqslant \Phi$, where Φ is the golden ratio. When I is normal.

Lemma 2

$\forall k \in [1, \Phi)$ class $\mathcal{I}_2^R(\mathrm{rank}\, P = 1, G = K_2)$ contains an instance with proportionality factor k, which is not normal.

Abnormality

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Lemma 3

Let $I \in \mathcal{I}_2^R(\mathrm{rank}\, P = 1, G = K_2)$ be an instance with proportionality factor $k \in [1, \Phi)$. When

$$R^*_{\max}(I) \leqslant \frac{4k^2 + 3k}{5k^2 + 2k - 1} \bar{R}.$$

Conclusion

Theorem 1

$$\alpha(\mathcal{I}_2^R(\mathrm{rank}\, P = 1, G = K_2)) = F(k) = \begin{cases} \frac{4k^2+3k}{5k^2+2k-1}, & k \in [1, \Phi), \\ 1, & k \geqslant \Phi. \end{cases}$$

For each $I \in \mathcal{I}_2^R(\mathrm{rank}\, P = 1, G = K_2)$ a feasible schedule with makespan $\leqslant F(k)\bar{R}$ can be built in linear time.

Proof

Straightforward from Lemmas 1,2,3.

Conclusion

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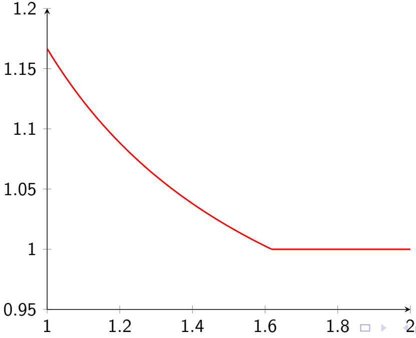

Conclusion

Theorem 2

$$\alpha(\mathcal{I}_2^R(\mathrm{rank}\, P = 1, G = K_3)) = F(k) = \begin{cases} \frac{4k^2+3k}{5k^2+2k-1}, & k \in [1, \Phi), \\ 1, & k \geqslant \Phi. \end{cases}$$

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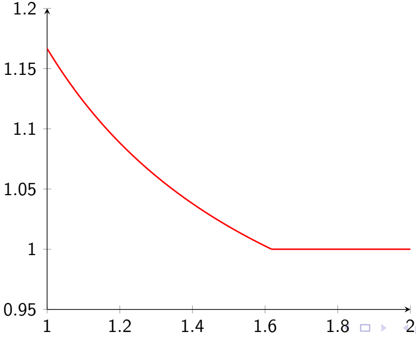

Open questions and plans

$$F(k) = \begin{cases} \frac{4k^2+3k}{5k^2+2k-1}, & k \in [1, \Phi), \\ 1, & k \geqslant \Phi. \end{cases}$$

1. Does similar result hold for $G = K_4$, $G = tree$, $G = cycle$, etc?
2. How does algorithmic complexity of the routing open shop on class $\mathcal{I}_2^R(\operatorname{rank} P = 1, G = K_2)$ depend on k:
   - $k = 1$: $\mathbb{NP}$-hard problem $RO2|j\text{-}prpt, G = K_2|R_{\max}$,
   - $k \geqslant \Phi$: solvable in linear time,
   - $k \in (1, \Phi)$: ???
3. Investigate $O3|\operatorname{rank} P = 1|C_{\max}$ and other shop scheduling problems with $\operatorname{rank} P = 1$.

Thank you for your attention!

Thanks!

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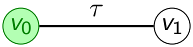
Thanks!

Questions?

$$RO2|\mathrm{rank}\, P = 1, G = K_2|R_{\max}.$$

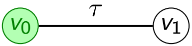


Processing times: $P = \begin{pmatrix} a_1 & \dots & a_n \\ b_1 & \dots & b_n \end{pmatrix} = \begin{pmatrix} kp_1 & \dots & kp_n \\ p_1 & \dots & p_n \end{pmatrix}$, without loss of generality $k \geqslant 1$.

$$\bar{R} = \max\{\ell_1 + 2\tau, d_{\max}(v_0), d_{\max}(v_1) + 2\tau\}.$$

The idea of proof for $G = K_2$

$$RO2|\mathrm{rank}\, P = 1, G = K_2|R_{\max}.$$



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$$\bar{R} = \max\{\ell_1 + 2\tau, d_{\max}(v_0), d_{\max}(v_1) + 2\tau\}.$$

The approach:

1. Preprocessing (instance reduction preserving $\bar{R}$)
2. Investigation of the simplified (reduced) instance

Preprocessing

1. Job aggregation (or grouping) preserving $\bar{R}$:
   - Grouping of jobs only from the same node v is allowed;
   - $\bar{R}$ is preserved, if total duration of jobs aggregated doesn't exceed $\bar{R} - 2\mathrm{dist}(v_0, v)$;
   - Using $\sum d_j = \ell_1 + \ell_2 \leqslant 2(\bar{R} - T^*)$ it is possible to group jobs in at most 3 groups per node;
   - Grouping doesn't violate machine loads, mode loads and proportionality factor k;
   - Any feasible schedule for the reduced instance can be treated as feasible schedule with same makespan for the initial instance.
2. Overloaded and superoverloaded nodes:
   - Node v is overloaded, if its load (total duration of jobs) is greater than $\bar{R} - 2\mathrm{dist}(v_0, v)$;
   - Any two-machine instance contains at most one overloaded node;
   - An instance is irreducible, if any possible job aggregation enlarges $\bar{R}$;
   - A node in a irreducible instance, containing three jobs, is superoverloaded.

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3 Some known facts:
- Any instance of $RO2|G = K_2|R_{\max}$ without overloaded nodes is normal [trivial]
- Any instance of $RO2|G = K_2|R_{\max}$ with a superoverloaded node is normal [Ch, Pyatkin 2020]
- Any instance of $RO2||R_{\max}$ with overloaded depot is normal [Ch 2021]

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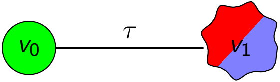
4 It is sufficient to consider instances of the following type:

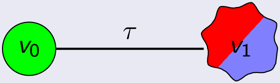


Node v_0 contains a single job J_1, node v_1 is overloaded and contains jobs J_2 and J_3.

Lemma 1

Let $I \in \mathcal{I}_2^R(\mathrm{rank}\, P = 1, G = K_2)$ is an instance with $k \geqslant \Phi$, where Φ is the golden ratio. When I is normal.

Proof



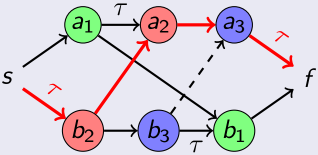
Node v_0 contains a single job J_1, node v_1 is overloaded and contains jobs J_2 and J_3.
Consider an irreducible instance with processing times

$$P = \left(\begin{array}{c|cc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{array} \right) = \left(\begin{array}{c|cc} kp_1 & kp_2 & kp_3 \\ p_1 & p_2 & p_3 \end{array} \right).$$

Without loss of generality assume $p_2 \geqslant p_3$.

Proof (continued)

Consider schedule S_1:



$$R_1 = R_{\max}(S_1) = b_2 + a_2 + a_3 + 2\tau = (1+k)p_2 + kp_3 + 2\tau.$$

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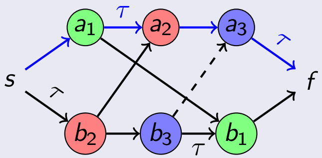

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Proof (continued)

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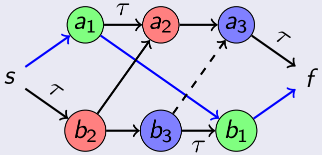

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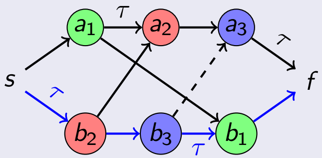

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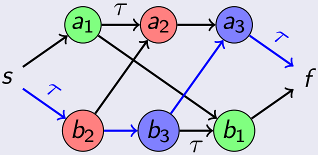

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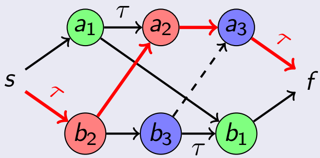

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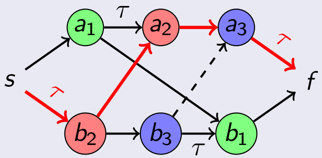
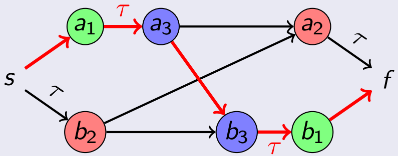


$$R_1 = R_{\max}(S_1) = b_2 + a_2 + a_3 + 2\tau = (1+k)p_2 + kp_3 + 2\tau.$$

Consider schedule S_2:



$$R_2 = R_{\max}(S_2) = a_1 + a_3 + b_3 + b_1 + 2\tau = (1+k)p_1 + (1+k)p_3 + 2\tau.$$

Let's prove, that at least one of schedules S_1 and S_2 is normal (for any instance complying with Lemma), or

$$\min\{R_1, R_2\} \leqslant \bar{R}.$$

Note that $\forall \lambda \in [0, 1]$

$$\min\{R_1, R_2\} \leqslant \lambda R_1 + (1 - \lambda) R_2.$$

On the other hand,

$$LB_1 = \ell_1 + 2\tau = k(p_1 + p_2 + p_3) + 2\tau \leqslant \bar{R},$$

$$LB_2 = k(p_1 + 2p_3) + 2\tau \leqslant LB_1 \leqslant \bar{R},$$

therefore $\forall \mu \in [0, 1]$ $\mu LB_1 + (1 - \mu) LB_2 \leqslant \bar{R}$.
It is sufficient to find such $\lambda, \mu \in [0, 1]$, that

$$\lambda R_1 + (1 - \lambda) R_2 \leqslant \mu LB_1 + (1 - \mu) LB_2.$$

Proof (finished)

$$R_1 = (1+k)p_2 + kp_3 + 2\tau,$$

$$R_2 = (1+k)(p_1+p_3) + 2\tau,$$

$$LB_1 = k(p_1+p_2+p_3) + 2\tau,$$

$$LB_2 = k(p_1+2p_3) + 2\tau.$$

Consider $\lambda = \frac{k-1}{k}$, $\mu = 1 - \frac{1}{k^2}$.

$$\lambda R_1 + (1-\lambda)R_2 = \frac{k+1}{k}p_1 + \frac{k^2-1}{k}p_2 + \frac{k^2+1}{k}p_3 + 2\tau,$$

$$\mu LB_1 + (1-\mu)LB_2 = kp_1 + \frac{k^2-1}{k}p_2 + \frac{k^2+1}{k}p_3 + 2\tau,$$

$$\lambda R_1 + (1-\lambda)R_2 \leqslant \mu LB_1 + (1-\mu)LB_2 \iff 1 + \frac{1}{k} \leqslant k \iff k^2 - k \geqslant 1.$$

Lemma 2

$\forall k \in [1, \Phi)$ class $\mathcal{I}_2^R(\mathrm{rank}\, P = 1, G = K_2)$ contains an instance with proportionality factor k, which is not normal.

Proof

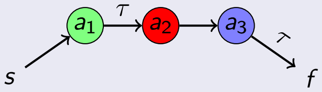
Consider instance $p_1 = k$, $p_2 = p_3 = k + 1$, $2\tau = 2k^2 - 1$.
For this instance we have
$\bar{R} = k(p_1 + p_2 + p_3) + 2\tau = 5k^2 + 2k - 1 < 4k^2 + 3k$.
Assume there exists a feasible schedule S with makespan

$$R_{\max}(S) < 4k^2 + 3k = p_1 + p_2 + p_3 + 4\tau.$$

Note that each machine travels once from v_0 to v_1 and back in S.
Without loss of generality M_1 performs operations in order $a_1 \to a_2 \to a_3$.

Proof (continued and finished)

$$p_1 = k, p_2 = p_3 = k + 1, 2\tau = 2k^2 - 1.$$



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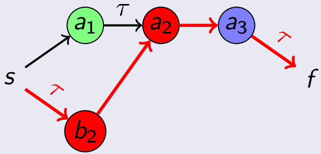
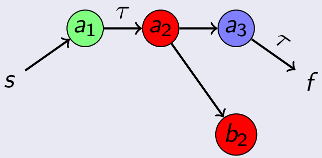


$$R_{\max}(S) \geqslant b_2+a_2+a_3+2\tau = k+1+k(k+1)+k(k+1)+2k_2-1 = 4k^2+3k.$$

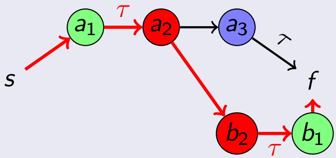
Proof (continued and finished)

$$p_1 = k, p_2 = p_3 = k + 1, 2\tau = 2k^2 - 1.$$



Proof (continued and finished)

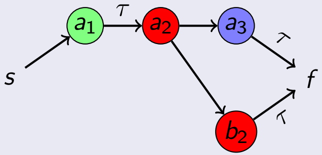
$$p_1 = k, p_2 = p_3 = k + 1, 2\tau = 2k^2 - 1.$$



$$R_{\max}(S) \geqslant a_1 + a_2 + b_2 + b_1 + 2\tau = (k+1)k + (k+1)^2 + 2k_2 - 1 = 4k^2 + 3k.$$

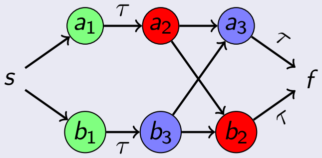
Proof (continued and finished)

$$p_1 = k, p_2 = p_3 = k + 1, 2\tau = 2k^2 - 1.$$



Proof (continued and finished)

$$p_1 = k, p_2 = p_3 = k + 1, 2\tau = 2k^2 - 1.$$



Proof (continued and finished)

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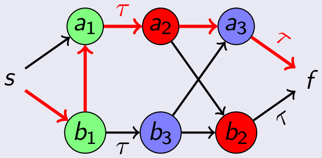
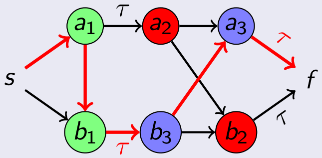


$$R_{\max}(S) \geqslant k + k \cdot (3k + 2)) + 2k^2 - 1 = 5k^2 + 3k - 1 \geqslant 4k^2 + 3k.$$

Proof (continued and finished)

$$p_1 = k, p_2 = p_3 = k+1, 2\tau = 2k^2 - 1.$$



$$R_{\max}(S) \geqslant a_1+b_1+b_3+a_3+2\tau = k^2+k+k+1+k(k+1)+2k^2-1 = 4k^2+3k.$$

Abnormality

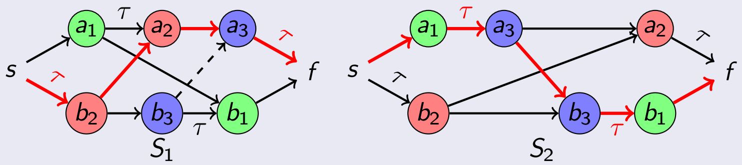
Lemma 3

Let $I \in \mathcal{I}_2^R(\mathrm{rank}\, P = 1, G = K_2)$ be an instance with proportionality factor $k \in [1, \Phi)$. When

$$R^*_{\max}(I) \leqslant \frac{4k^2 + 3k}{5k^2 + 2k - 1}\bar{R}.$$

Proof

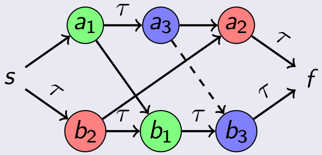
Consider the following three schedules:



$$R_1 = R_{\max}(S_1) = b_2 + a_2 + a_3 + 2\tau = (1 + k)p_2 + kp_3 + 2\tau.$$

$$R_2 = R_{\max}(S_2) = a_1 + a_3 + b_3 + b_1 + 2\tau = (1 + k)p_1 + (1 + k)p_3 + 2\tau.$$

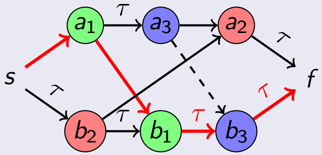
$$R_1 = (1+k)p_2 + kp_3 + 2\tau, R_2 = (1+k)p_1 + (1+k)p_3 + 2\tau.$$



S_3

Proof (continued)

$$R_1 = (1+k)p_2 + kp_3 + 2\tau, R_2 = (1+k)p_1 + (1+k)p_3 + 2\tau.$$



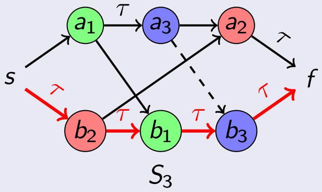
S_3

Case 1:

$$R_3 = R_{\max}(S_3) = a_1 + b_1 + b_3 + 2\tau = (k+1)p_1 + p_3 + 2\tau,$$

$$R_1 + R_3 = (k+1)(p_1 + p_2 + p_3) + 4\tau = \ell_1 + \ell_2 + 4\tau \leqslant 2\bar{R}.$$

$$R_1 = (1+k)p_2 + kp_3 + 2\tau, R_2 = (1+k)p_1 + (1+k)p_3 + 2\tau.$$



Case 2:

$$R_3 = R_{\max}(S_3) = b_2 + b_1 + b_3 + 4\tau = p_1 + p_2 + p_3 + 4\tau.$$

Proof, Case 2

$$R_1 = (1+k)p_2 + kp_3 + 2\tau,$$

$$R_2 = (1+k)p_1 + (1+k)p_3 + 2\tau,$$

$$R_3 = R_{\max}(S_3) = b_2 + b_1 + b_3 + 4\tau = p_1 + p_2 + p_3 + 4\tau.$$

Let's prove that $\min\{R_1, R_2, R_3\} \leqslant \frac{4k^2+3k}{5k^2+2k-1}\bar{R}$.

Proof, Case 2

$$R_1 = (1+k)p_2 + kp_3 + 2\tau,$$

$$R_2 = (1+k)p_1 + (1+k)p_3 + 2\tau,$$

$$R_3 = R_{\max}(S_3) = b_2 + b_1 + b_3 + 4\tau = p_1 + p_2 + p_3 + 4\tau.$$

Let's prove that $\min\{R_1, R_2, R_3\} \leqslant \frac{4k^2+3k}{5k^2+2k-1}\bar{R}$.

$$LB_1 = k(p_1 + p_2 + p_3) + 2\tau,\ LB_2 = k(p_1 + 2p_3) + 2\tau.$$

It is sufficient to find such $\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2$, that

1. $\lambda_1 R_1 + \lambda_2 R_2 + \lambda_3 R_3 \leqslant \mu_1 LB_1 + \mu_2 LB_2$,
2. $(\lambda_1 + \lambda_2 + \lambda_3)(4k^2 + 3k) \leqslant (\mu_1 + \mu_2)(5k^2 + 2k - 1)$.

Proof, Case 2

$$R_1 = (1+k)p_2 + kp_3 + 2\tau,$$

$$R_2 = (1+k)p_1 + (1+k)p_3 + 2\tau,$$

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Let's prove that $\min\{R_1, R_2, R_3\} \leqslant \frac{4k^2+3k}{5k^2+2k-1}\bar{R}$.

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2. $(\lambda_1 + \lambda_2 + \lambda_3)(4k^2 + 3k) \leqslant (\mu_1 + \mu_2)(5k^2 + 2k - 1)$.

$$\lambda_1 = \frac{2k^2 + k - 1}{k^2},\ \lambda_2 = \frac{4k^2 - 1}{k^2},\ \lambda_3 = \frac{-k^2 + k + 1}{k^2},$$

$$\mu_1 = \lambda_2 + 2\lambda_3 = \frac{2k^2 + 2k + 1}{k^2},\ \mu_2 = \lambda_1 = \frac{2k^2 + k - 1}{k^2}.$$