# Axiomatization of Independence Systems by Map Functions

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# Independence System

$U$ – nonempty finite set, $\mathbf{A} \subseteq 2^{U}$ – nonempty indexed family of subsets with an *independence axiom*

$$A_1 \in \mathbf{A},\ A_2 \subseteq A_1 \Rightarrow A_2 \in \mathbf{A}.$$

**A** – *independent subsets*.

A set $\mathbf{D} = \mathbf{2}^{\mathbf{U}} \setminus \mathbf{A}$ is satisfied

$$D_1 \in \mathbf{D},\ D_1 \subseteq D_2 \Rightarrow D_2 \in \mathbf{D}.$$

**D** – *dependent subsets*.

**B** – *bases* or maximal independent subsets.

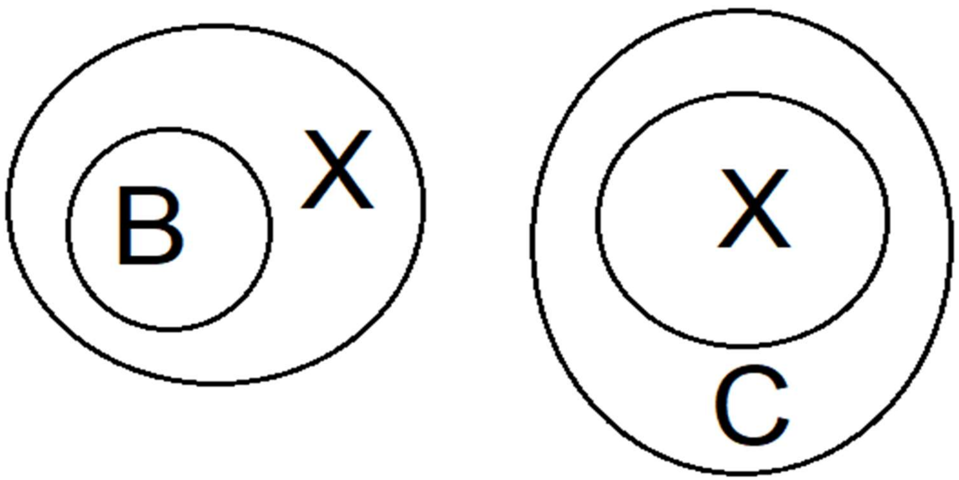
**C** – *circuities* or minimal dependent subsets.

$S = (U, \mathbf{A})$, $S = (U, \mathbf{B})$, $S = (U, \mathbf{C})$ and $S = (U, \mathbf{D})$ are equals *independence systems*.

# Independence System

For any subset $X \subseteq U$ let $\mathbf{B_X}$ be a set of all maximal independent *subsets* of $X$ (i.e. $\mathbf{B_X}$ is *bases* of $X$).

For any subset $X \subseteq U$ let $\mathbf{C_X}$ be a set of all minimal dependent *supersets* of $X$ (i.e. $\mathbf{C_X}$ is *circuities* of $X$).



# Graph Independence System

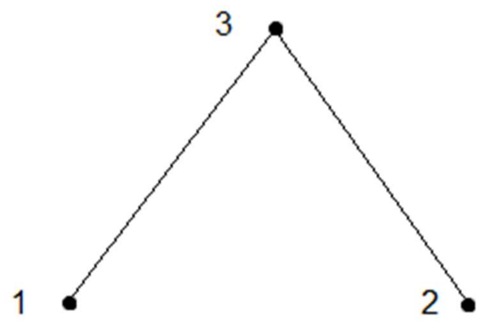
Example. Let $G = (V, E)$ be a graph. A subset $I \subseteq V$ is an *independent set* if $uv \notin E$ for any $u,v \in I$. A system $S = (V, \mathbf{A})$ where $\mathbf{A}$ is a family of all independent subsets in $G$ is an independence system.

$\mathbf{A} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}\}$

$\mathbf{D} = \{\{1,3\}, \{2,3\}, \{1,2,3\}\}$

$\mathbf{B} = \{\{3\}, \{1,2\}\}$

$\mathbf{C} = \{\{1,3\}, \{2,3\}\}$

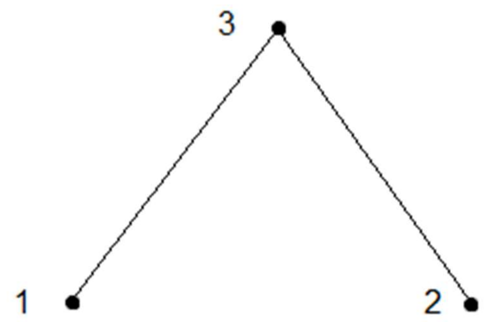


# Optimization Problems For Independence Systems

Many optimization problems are equivalent to a problem of finding a *base of maximal weight* or a *circuity of minimal weight*.

MAXIMAL INDEPENDENT SET PROBLEM: find maximal independent set in a graph.

For example, {1,2} - maximal independent set.




It is easy to see that {1,2} - base of maximal weight (let all weights be equal to 1).

This problem is NP-hard (in general).

# Graph Matroid

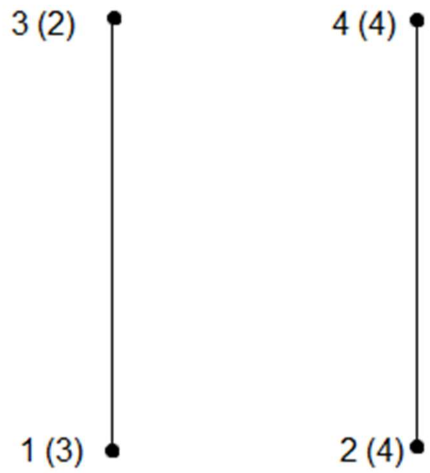
*Matroid* is an independence system $S = (U, \mathbf{A})$ if all bases of any $W \subseteq U$ have an equal cardinality.

$\mathbf{B} = \{\{1,2\}, \{1,4\}, \{2,3\}, \{3,4\}\}$

$\mathbf{C} = \{\{1,3\}, \{2,4\}\}$

$\{1,2\}$ and $\{1,4\}$ are bases of maximal weights.

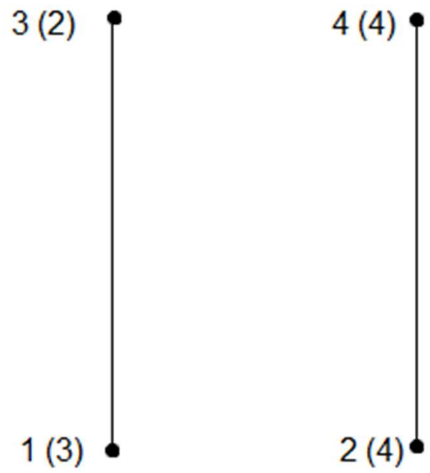
A graph independence system is a matroid if and only if the graph is a cluster graph.



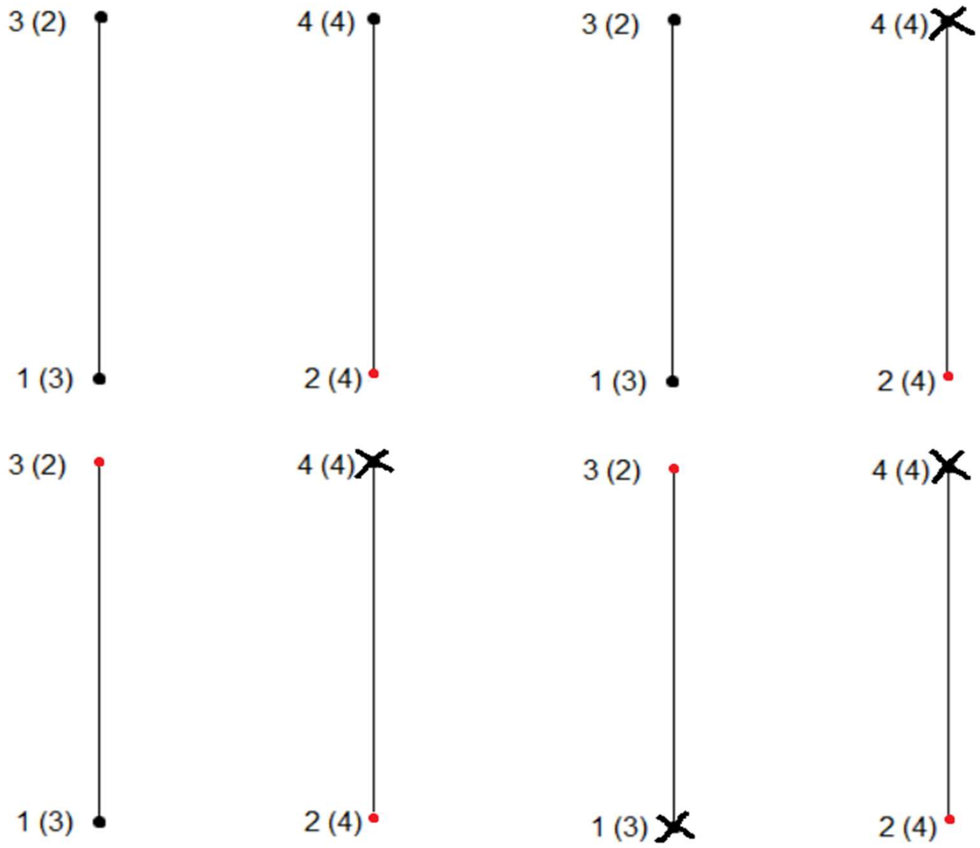
# Maximal Weighted Independent Set

We can find a base of maximal weight by a *greedy* algorithm in a matroid if an objective function is *additive* (Rado-Edmonds).

In general, we can't find optimal solution by a greedy algorithm, but we can use it for approximation algorithms.



# Maximal Weighted Independent Set



# Matroid and Rank Function

$M = (U, \mathbf{A})$ is a matroid. $\mathbf{r}$: $\mathbf{2}^{\mathbf{U}} \to \mathbb{Z}_+$ is a *rank function* that for each $X \in U$ maps $\mathbf{r}(X)$ – cardinality of any base $B \subseteq X$. Here

$$\mathbf{A} = \{A \subseteq U \mid \mathbf{r}(A) = |A|\} \ (1).$$

**Theorem.**

1) $M = (U, \mathbf{A})$ is a matroid. Then a rank function $\mathbf{r}$: $\mathbf{2}^{\mathbf{U}} \to \mathbb{Z}_+$ defined with

$$\mathbf{r}(X) = \max\{|B|: B \subseteq X, B \in \mathbf{B}_{\mathbf{X}}\} \ (2).$$

for each $X, Y \subseteq U$ satisfies:

(r1) $\mathbf{r}(X) \leq |X|$,

(r2) $X \subseteq Y \Rightarrow \mathbf{r}(X) \leq \mathbf{r}(Y)$ (monotonic),

(r3) $\mathbf{r}(X \cup Y) + \mathbf{r}(X \cap Y) \leq \mathbf{r}(X) + \mathbf{r}(Y)$ (submodular).

2) $\mathbf{r}$ is a rank function (2) and it satisfies (r1)-(r3). Then $\mathbf{A}$ defines with (1) is an indexed family of independent subsets of a matroid.

# Rank Function For Graph Matroid

Let's calculate rank for $M = (V, \mathbf{A})$, where $V = \{1, 2, 3\}$.

$\mathbf{B_V} = \{\{1,2\}, \{1,3\}\}$

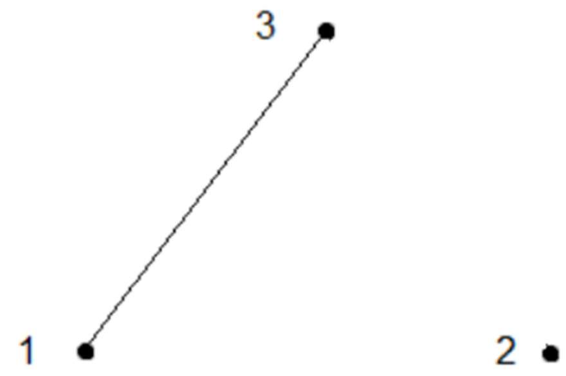
$\mathbf{r}(\underline{\emptyset}) = 0;$

$\mathbf{r}(\underline{\{1\}}) = \mathbf{r}(\underline{\{2\}}) = \mathbf{r}(\underline{\{3\}}) = 1;$

$\mathbf{r}(\underline{\{1,2\}}) = 2;$

$\mathbf{r}(\{1,3\}) = 1;$

$\mathbf{r}(\underline{\{2,3\}}) = 2;$
$\mathbf{r}(\{1,2,3\}) = 2.$



# Independence System and Rank Functions

For an independence system $S = (U, \mathbf{A})$ we define two functions.

$$\mathbf{r_u}(X) = \max\{|B| : B \subseteq X, B \in \mathbf{B_X}\} \text{ - upper rank}$$

$$\mathbf{r_l}(X) = \min\{|B| : B \subseteq X, B \in \mathbf{B_X}\} \text{ - lower rank}$$

**Theorem.**

Let $S = (U, \mathbf{A})$ be an independence system. Then the following conditions are equivalent:

1. $S$ is a matroid;
2. $\mathbf{r_u}$ and $\mathbf{r_l}$ are equals;
3. $\mathbf{r_u}$ is submodular.

# Independence System and Upper Rank Function

If $S = (U, \mathbf{A})$ is not a matroid then $\mathbf{r_u}$ satisfies (r1)-(r2) and doesn't satisfy (r3).

$\mathbf{r_u}(\underline{\emptyset}) = 0;$

$\mathbf{r_u}(\underline{\{1\}}) = \mathbf{r_u}(\underline{\{2\}}) = \mathbf{r_u}(\underline{\{3\}}) = 1;$

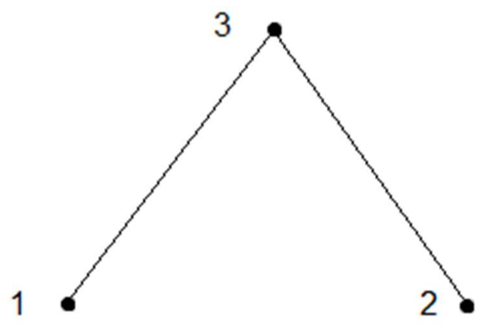
$\mathbf{r_u}(\underline{\{1,2\}}) = 2;$

$\mathbf{r_u}(\{1,3\}) = 1;$

$\mathbf{r_u}(\{2,3\}) = 1;$

$\mathbf{r_u}(\{1,2,3\}) = 2.$

$\mathbf{r_u}(\{1,3\}) + \mathbf{r_u}(\{2,3\}) < \mathbf{r_u}(\{1,2,3\}) + \mathbf{r_u}(\{3\}).$



# Independence System and Upper Rank Function

$S = (U, \mathbf{A})$ is an independence system. $\mathbf{r_u}: \mathbf{2^U} \to \mathbb{Z}_+$ is an *upper rank function*. Here

$$\mathbf{A} = \{A \subseteq U \mid \mathbf{r_u}(A) = |A|\} \ (1).$$

**Theorem.**

1) $S = (U, \mathbf{A})$ is an independence system. Then an upper rank function $\mathbf{r_u}: \mathbf{2^U} \to \mathbb{Z}_+$ defined with

$$\mathbf{r_u}(X) = \max\{|B| : B \subseteq X, B \in \mathbf{B_X}\} \ (2).$$

for each $X, Y \subseteq U$ satisfies:

(r1) $\mathbf{r_u}(X) \le |X|$,

(r2) $X \subseteq Y \Rightarrow \mathbf{r_u}(X) \le \mathbf{r_u}(Y)$,

(r3) $\mathbf{r_u}(X \cup Y) \le \mathbf{r_u}(X) + \mathbf{r_u}(Y)$.

2) $\mathbf{r_u}$ is an upper rank function (2) and it satisfies (r1)-(r3). Then $\mathbf{A}$ defines with (1) is an indexed family of independent subsets of an independence system.

# Comatroid

*Comatroid* is an independence system $S = (U, \mathbf{D})$ if all circuities of any $W \subseteq U$ have an equal cardinality.

Example. Let $G = (V, E)$ be a graph. A subset $I \subseteq V$ is a *cover set* of $V$ if for each $uv \in E$: $u \in V$ or $v \in V$ or $u,v \in V$. A system $S = (V, \mathbf{D})$ where $\mathbf{D}$ is a family of all cover sets in $G$ is an independence system.

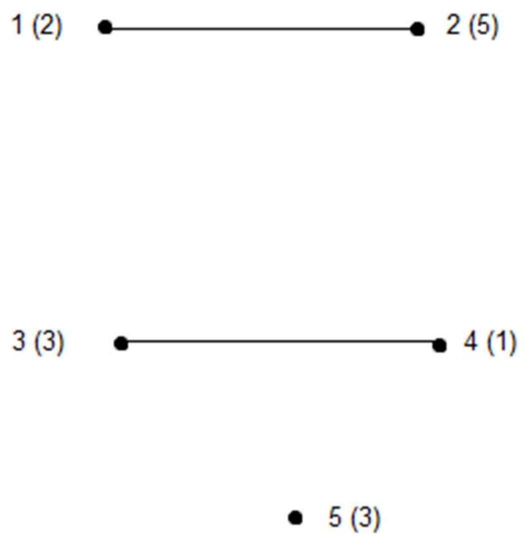
# Graph Comatroid

A graph independent system is a comatroid if and only if the graph is a cluster graph.

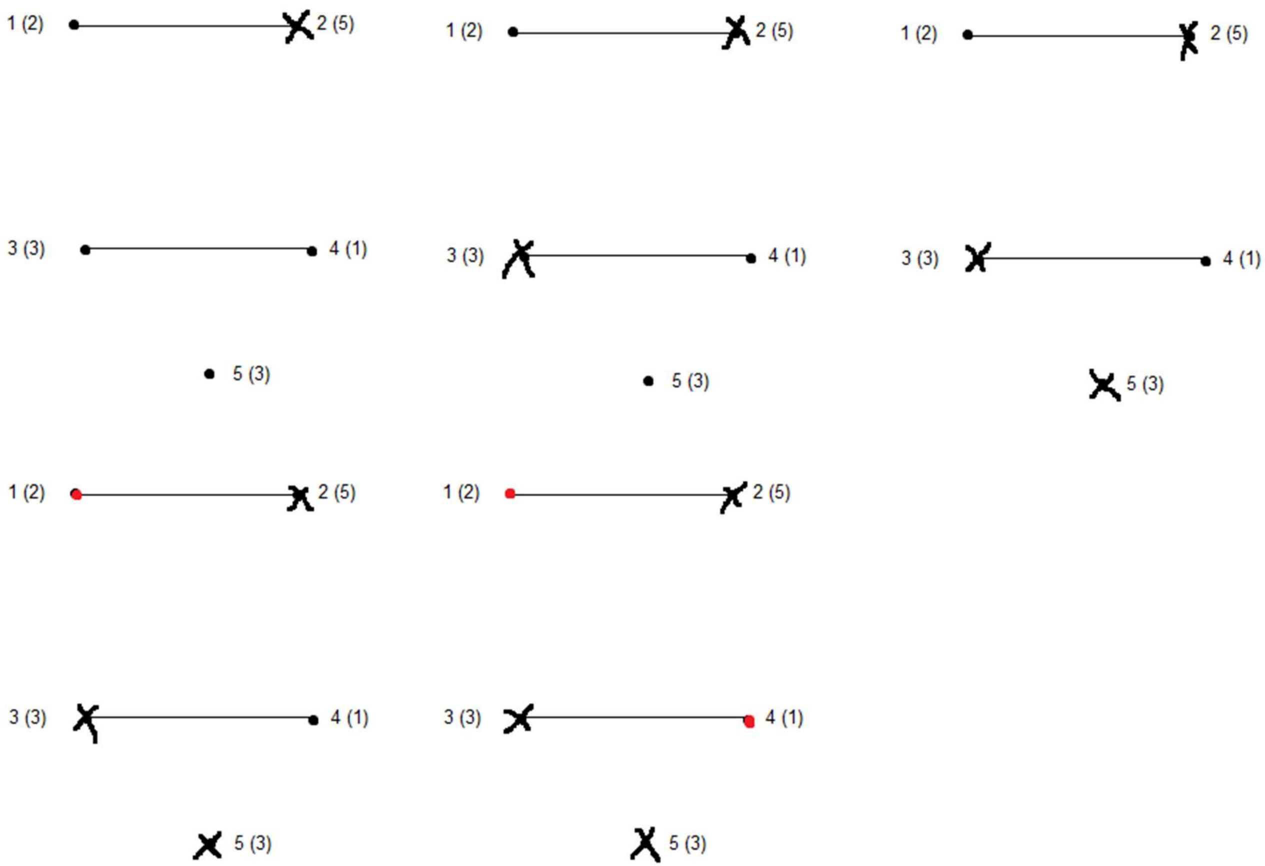
**C** = {{1,3}, {1,4}, {2,3}, {2,4}}

We can find a circuity of minimal weight by a *reverse greedy* algorithm in a comatroid if an objective function is *additive* (Rado-Edmonds).

In general, we can't find optimal solution by the reverse greedy algorithm, but we can use it for approximation algorithms.



# Graph Comatroid



# Comatroid and Girth Function

$CM = (U, \mathbf{A})$ is a comatroid. $\mathbf{g}: \mathbf{2}^{\mathbf{U}} \rightarrow \mathbb{Z}_+$ is a *girth function* that for each $X \in U$ maps $\mathbf{g}(X)$ – cardinality of any circus $X \subseteq C$. Here

$$\mathbf{D} = \{D \subseteq U \mid \mathbf{g}(D) = |D|\} \ (1).$$

**Theorem.**

1) $CM = (U, \mathbf{D})$ is a comatroid. Then a girth function $\mathbf{g}: 2^{\mathrm{U}} \rightarrow \mathbb{Z}_+$ defined with

$$\mathbf{g}(X) = \min\{|C| : X \subseteq C, C \in \mathbf{C_X}\} \ (2).$$

for each $X, Y \subseteq U$ satisfies:

(g1) $\mathbf{g}(X) \geq |X|$,

(g2) $X \subseteq Y \Rightarrow \mathbf{g}(X) \leq \mathbf{g}(Y)$ (monotonic),

(g3) $\mathbf{g}(X \cup Y) + \mathbf{g}(X \cap Y) \geq \mathbf{g}(X) + \mathbf{g}(Y)$ (supermodular).

2) $\mathbf{g}$ is a girth function (2) and it satisfies (g1)-(g3). Then $\mathbf{D}$ defines with (1) is an indexed family of dependent subsets of a comatroid.

# Plans

1. Independence system by the grid function.
2. Independence system by the interior operator.