Definitions of Hereditary Systems in Terms of Rank and Girth Functions

Aleksandr Morshinin

Sobolev Institute of Mathematics, SB RAS

The research was supported by Russian Science Foundation grant N 22-71-10015-П,
https://rscf.ru/en/project/22-71-10015-П

Introduction

Let U be a nonempty finite set, and let $\mathcal{A} \subseteq 2^U$ be a nonempty family of its subsets satisfying the hereditary axiom:

$$(A1) \quad J \in \mathcal{A},\ I \subseteq J \Rightarrow I \in \mathcal{A}.$$

Note that the elements of the family $\mathcal{D} = 2^U \setminus \mathcal{A}$ satisfy the following “upward” hereditary property:

$$(D1) \quad I \in \mathcal{D},\ I \subseteq J \Rightarrow J \in \mathcal{D}.$$

A *hereditary system* $\mathbf{S}$ on the set U is a partition of the family 2^U of all subsets of U into two disjoint families $\mathcal{A}$ and $\mathcal{D}$.
A *basis* of a set X is any inclusion-maximal independent subset of X.
A *circuit* of a set X is any inclusion-minimal dependent superset of X.

Matroids and Comatroids

A *matroid* on a set U can be defined as a hereditary system $\mathbf{M} = (U, \mathcal{A})$ in which all bases of every set $X \subseteq U$ have the same cardinality.
Equivalently, a matroid additionally satisfies the property

$$(A2) \quad I, J \in \mathcal{A},\ |J| = |I| + 1 \Rightarrow \exists a \in J \setminus I : I \cup \{a\} \in \mathcal{A}.$$

A *comatroid* on a set U can be defined as a hereditary system $\mathbf{K} = (U, \mathcal{D})$ in which all circuits of every set $X \subseteq U$ have the same cardinality.
Equivalently, a comatroid additionally satisfies the property

$$(D2) \quad I, J \in \mathcal{D},\ |J| = |I| + 1 \Rightarrow \exists d \in J \setminus I : J \setminus \{d\} \in \mathcal{D}.$$

Hereditary systems, matroids, and comatroids arise in such areas of mathematics as linear algebra, graph theory, combinatorial geometries, and others.

Combinatorial Optimization

Many optimization problems can be formulated in the form

$$\max\{f(X) : X \in \mathcal{A}\} \qquad \text{or} \qquad \min\{f(X) : X \in \mathcal{D}\},$$

where $f : 2^U \to \mathbb{R}_+$ is a monotone nonnegative set function, and $\mathcal{A}$ and $\mathcal{D}$ are the families of independent and dependent sets of some hereditary system.

Examples include the minimum and maximum spanning tree problem, the p-median problem, the maximum independent set problem, the minimum vertex cover problem, the correlation clustering problem, and so on.

Hereditary System of a Graph

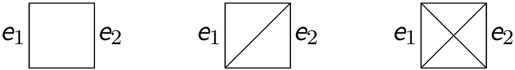
Let $G = (V, E)$ be a simple graph. A set $X \subseteq V$ is called *independent* if no two vertices from X are adjacent. Then

$$\mathbf{S}_G = (V, \mathcal{A}_G)$$

is a hereditary system, where the family $\mathcal{A}_G$ consists of all independent vertex sets of the graph G. The circuits of this system are in one-to-one correspondence with the edges of G.

It is easy to see that the hereditary system $\mathbf{S}_G$ is a matroid if and only if G is a cluster graph. This matroid is called the *partition matroid*.

The hereditary system $\mathbf{S}_G$ is a comatroid if and only if every pair of nonadjacent edges e_1, e_2 of the graph G belongs to a cycle of length 4.



Rank and Girth of a Hereditary System

For any hereditary system $\mathbf{S} = (U, \mathcal{A})$, one can define the *rank function*:

$$r(X) = \max\{|A| : A \subseteq X,\ A \in \mathcal{A}\} \tag{1}$$

Clearly,

$$\mathcal{A} = \{A \subseteq U : r(A) = |A|\}. \tag{2}$$

Similarly, one can define the *girth function*:

$$g(X) = \min\{|D| : D \supseteq X,\ D \in \mathcal{D}\}. \tag{3}$$

Clearly,

$$\mathcal{D} = \{D \subseteq U : g(D) = |D|\}. \tag{4}$$

Rank of a Matroid

Theorem

1) Let $\mathbf{M} = (U, \mathcal{A})$ *be a matroid. Then the rank function* (1) *satisfies the following conditions for all* $X, Y \subseteq U$:

(r1) $0 \le r(X) \le |X|$,

(r2) $r(\emptyset) = 0$,

(r3) $r(X) \le r(X \cup \{u\}) \le r(X) + 1$ *for all* $u \notin X$,

(r4) *if* $X \subseteq Y$, *then* $r(X) \le r(Y)$,

(r5) $r(X \cup Y) + r(X \cap Y) \le r(X) + r(Y)$,

(r6) *if* $r(X \cup \{u\}) = r(X \cup \{v\}) = r(X)$ *for some* $u, v \notin X$, *then* $r(X \cup \{u, v\}) = r(X)$,

and equality (2) *holds.*

2) Let U *be a nonempty finite set, and let* $r : 2^U \to \mathbb{Z}_+$ *be a set function satisfying either conditions* (r2), (r3), (r6) *or conditions* (r1), (r4), (r5). *Then* $\mathbf{M} = (U, \mathcal{A})$ *is a matroid, where the family* $\mathcal{A} \subseteq 2^U$ *is defined by* (2), *and equality* (1) *holds for all* $X \subseteq U$.

Rank of a Matroid

Thus, a *matroid* is a pair $\mathbf{M} = (U, r)$, where U is a nonempty finite set and $r : 2^U \to \mathbb{Z}_+$ is a set function satisfying one of the following sets of conditions: for all $X, Y \subseteq U$

1. $(r1)$, $(r4)$, $(r5)$,
2. $(r2)$, $(r3)$, $(r6)$.

Girth of a Comatroid

Theorem

1) Let $\mathbf{K} = (U, \mathcal{D})$ *be a comatroid. Then the girth function of a comatroid* (3) *satisfies the following conditions for all* $X, Y \subseteq U$*:*

(g1) $|U| \geq g(X) \geq |X|$,

(g2) $g(U) = |U|$,

(g3) $g(X) \geq g(X \setminus \{u\}) \geq g(X) - 1$ *for all* $u \in X$,

(g4) *if* $X \subseteq Y$, *then* $g(X) \leq g(Y)$,

(g5) $g(X \cup Y) + g(X \cap Y) \geq g(X) + g(Y)$,

(g6) *if* $g(X \setminus \{u\}) = g(X \setminus \{v\}) = g(X)$ *for some* $u, v \in X$, *then* $g(X \setminus \{u, v\}) = g(X)$,

and equality (4) *holds.*

2) Let U *be a nonempty finite set, and let* $g : 2^U \to \mathbb{Z}_+$ *be a set function satisfying either conditions* (g2), (g3), (g6) *or conditions* (g1), (g4), (g5). *Then* $\mathbf{K} = (U, \mathcal{D})$ *is a comatroid, where the family* $\mathcal{D} \subseteq 2^U$ *is defined by* (4), *and equality* (3) *holds for all* $X \subseteq U$.

Girth of a Comatroid

A *comatroid* is a pair $\mathbf{K} = (U, g)$, where U is a nonempty finite set and $g : 2^U \to \mathbb{Z}_+$ is a set function satisfying one of the following sets of conditions: for all $X, Y \subseteq U$

1. $(g1)$, $(g4)$, $(g5)$,
2. $(g2)$, $(g3)$, $(g6)$.

Rank and Girth of a Hereditary System

In the case of an arbitrary hereditary system, for which conditions $(A2)$ and $(D2)$ may fail, the rank and girth functions r and g must satisfy conditions different from those for a matroid.

In our previous work, we established the following equivalent definition.

A *hereditary system* is a pair $\mathbf{S} = (U, r)$, where U is a nonempty finite set and $r : 2^U \to \mathbb{Z}_+$ is a set function satisfying the following conditions for every $X \subseteq U$:

$(r'1)$ $0 \leq r(X) \leq |X|$,

$(r'5)$ $r(X) = \max\{|A| : A \subseteq X \text{ such that } r(A) = |A|\}$,

$(r'7)$ if $X \subseteq Y$, then $r(X) \leq r(Y)$,

$(r'8)$ $r(X \cup Y) \leq r(X) + r(Y)$.

It can be shown that condition $(r'7)$ is redundant.

Rank of a Hereditary System

Theorem

Let $\mathbf{S} = (U, \mathcal{A})$ be a hereditary system. Then the rank function (1) *satisfies the following conditions for all $X \subseteq U$:*

($r'1$) $0 \leq r(X) \leq |X|$,

($r'2$) $r(\emptyset) = 0$,

($r'3$) $r(X) \leq r(X \cup \{u\}) \leq r(X) + 1$ *for all* $u \notin X$,

($r'4$) *if* $r(X) = |X|$, *then* $r(X \setminus \{u\}) = |X| - 1$ *for all* $u \in X$,

($r'5$) $r(X) = \max\{|A| : A \subseteq X \text{ such that } r(A) = |A|\}$,

($r'6$) *if* $A \subseteq X \subseteq U$, $u \in U \setminus X$ *are such that* $r(X) = r(A) = |A|$, *but* $r(X \cup \{u\}) = r(X) + 1$, *then there exists* $A' \subseteq X \cup \{u\}$ *such that* $r(A') = |A'| > |A|$,

and equality (2) *holds.*

Rank of a Hereditary System

Theorem

Let U be a nonempty finite set, and let $r : 2^U \to \mathbb{Z}_+$ be a set function satisfying simultaneously:

1. *condition* $(r'1)$ *or condition* $(r'2)$,
2. *conditions* $(r'1)$, $(r'3)$ *or condition* $(r'4)$,
3. *condition* $(r'5)$ *or conditions* $(r'1)$, $(r'3)$, $(r'6)$.

Then $\mathbf{S} = (U, \mathcal{A})$ is a hereditary system, where $\mathcal{A} \subseteq 2^U$ is defined by (2)*, and equality* (1) *holds for all $X \subseteq U$.*

Rank of a Hereditary System

Condition $(r'1)$ follows from $(r'2)$ and $(r'3)$. Condition $(r'4)$ is redundant whenever $(r'1)$ and $(r'3)$ hold.

A *hereditary system* is a pair $\mathbf{S} = (U, r)$, where U is a nonempty finite set and $r : 2^U \to \mathbb{Z}_+$ is a set function satisfying one of the following sets of conditions for all $X \subseteq U$:

1. $(r'1)$, $(r'3)$, $(r'5)$,
2. $(r'1)$, $(r'3)$, $(r'6)$,
3. $(r'1)$, $(r'4)$, $(r'5)$,
4. $(r'2)$, $(r'3)$, $(r'5)$,
5. $(r'2)$, $(r'3)$, $(r'6)$,
6. $(r'2)$, $(r'4)$, $(r'5)$.

Girth of a Hereditary System

Theorem

Let $\mathbf{S} = (U, \mathcal{D})$ *be a hereditary system distinct from the free matroid. Then the girth function of a hereditary system* (3) *satisfies the following conditions for all* $X \subseteq U$*:*

($g'1$) $|U| \geq g(X) \geq |X|$,

($g'2$) $g(U) = |U|$,

($g'3$) $g(X) \geq g(X \setminus \{u\}) \geq g(X) - 1$ *for all* $u \in X$,

($g'4$) *if* $g(X) = |X|$, *then* $g(X \cup \{u\}) = |X| + 1$ *for all* $u \in U \setminus X$,

($g'5$) $g(X) = \min\{|D| : D \supseteq X \text{ such that } g(D) = |D|\}$,

($g'6$) *if* $X \subseteq D \subseteq U$, $u \in X$ *are such that* $g(X) = g(D) = |D|$, *but* $g(X \setminus \{u\}) = g(D) - 1$, *then there exists* $D' \supseteq X \setminus \{u\}$ *such that* $g(D') = |D'| < |D|$,

and equality (4) *holds.*

Girth of a Hereditary System

Theorem

Let U be a nonempty finite set, and let $g : 2^U \to \mathbb{Z}_+$ be a set function satisfying simultaneously:

1. *condition* $(g'1)$ *or condition* $(g'2)$*;*
2. *conditions* $(g'1)$, $(g'3)$ *or condition* $(g'4)$*;*
3. *condition* $(g'5)$ *or conditions* $(g'1)$, $(g'3)$, $(g'6)$.

Then $\mathbf{S} = (U, \mathcal{D})$ *is a hereditary system distinct from the free matroid, where the family* $\mathcal{D} \subseteq 2^U$ *is defined by* (4), *and equality* (3) *holds for all* $X \subseteq U$.

Girth of a Hereditary System

A *hereditary system* is a pair $\mathbf{S} = (U, g)$, where U is a nonempty finite set and $g : 2^U \to \mathbb{Z}_+$ is a set function satisfying one of the following sets of conditions for all $X \subseteq U$:

1. $(g'1)$, $(g'3)$, $(g'5)$,
2. $(g'1)$, $(g'3)$, $(g'6)$,
3. $(g'1)$, $(g'4)$, $(g'5)$,
4. $(g'2)$, $(g'3)$, $(g'5)$,
5. $(g'2)$, $(g'3)$, $(g'6)$,
6. $(g'2)$, $(g'4)$, $(g'5)$.

Thank you for your attention!