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GROUPS WITH EXPONENTS I. FUNDAMENTALS OF THE THEORY AND TENSOR COMPLETIONS[†)]

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We revise R. Lyndon's notion of group with exponents [1]. The advantage of the revised notion is that, in the case of abelian groups, it coincides with the notion of a module over a ring. Meanwhile, the abelian groups with exponents in the sense of Lyndon form a substantially wider class. In what follows we introduce basic notions of the theory of groups with exponents; in particular, we present the key construction in the category of groups with exponents, that of tensor completion.

The main results of the article are exposed in [2]; the notions of free A-group and free product of A-groups can be found in [3].

§ 1. Basic Notions of the Theory of Groups with Exponents

1.1. Definition of a group with exponents. Let A be an associative ring with unity and let G be a group. We denote the result of the action of an $\alpha \in A$ on a $g \in G$ by g^α. Consider the following axioms:

$$g^1 = g, \quad g^0 = 1, \quad 1^\alpha = 1, \tag{1}$$
$$g^{\alpha+\beta} = g^\alpha g^\beta, \quad g^{\alpha\beta} = (g^\alpha)^\beta, \tag{2}$$
$$(h^{-1}gh)^\alpha = h^{-1}g^\alpha h, \tag{3}$$
$$[g,h] = 1 \Rightarrow (gh)^\alpha = g^\alpha h^\alpha. \tag{4}$$

DEFINITION 1. A group G is called a *group with exponents over A in the sense of Lyndon* if A acts on G and the action satisfies axioms (1)–(3).

DEFINITION 2. A group G is called a *group with exponents over A* or an *A-group* if A acts on G and the action satisfies axioms (1)–(4).

We let $\mathfrak{L}_A$ denote the class of all groups with exponents over A in the sense of Lyndon and let $\mathfrak{M}_A$ denote the class of all A-groups. Clearly, $\mathfrak{L}_A \supseteq \mathfrak{M}_A$.

PROPERTY 1. *Every abelian A-group is an A-module and vice versa.*

Meanwhile, there are abelian groups with exponents over A in the sense of Lyndon which are not A-modules.

EXAMPLE 1 [1]. Let θ be a nonidentical automorphism of a ring A and let M be a free A-module with a base $\{x, y\}$. Consider a new action $\odot$ of A on M:

$$z \odot \alpha = \begin{cases} z \cdot \theta(\alpha) & \text{if} \quad z \in xA \cup yA, \\ z \cdot \alpha & \text{if} \quad z \notin xA \cup yA. \end{cases}$$

The action satisfies axioms (1)–(3); however, if $\alpha_0 \neq \theta(\alpha_0)$ then

$$(x+y) \odot \alpha_0 = (x+y)\alpha_0 \neq (x+y)\theta(\alpha_0) = x \odot \alpha_0 + y \odot \alpha_0,$$

violating axiom (4).

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Corollary 1. *$\mathfrak{M}_A$ is a proper subclass of $\mathfrak{L}_A$.*

PROPERTY 2. *If $G \in \mathfrak{L}_A$ then $[x,y]=1 \Rightarrow [x^\alpha, y^\beta]=1$ for arbitrary $\alpha, \beta \in A$ and $x,y \in G$.*
□ We have $yx^\alpha = yx^\alpha y^{-1} y = (yxy^{-1})^\alpha y = x^\alpha y$. Hence

$$x^\alpha y^\beta = x^\alpha y^\beta x^{-\alpha} x^\alpha = (x^\alpha y x^{-\alpha})^\beta x^\alpha = y^\beta x^\alpha. \ \square$$

Most of the natural examples of groups with exponents belong to $\mathfrak{M}_A$:
every group is a $\mathbb{Z}$-group;
every group in $\mathfrak{L}_{\mathbb{Q}}$ with unique roots is a $\mathbb{Q}$-group;
every group of period m is a $\mathbb{Z}/m\mathbb{Z}$-group;
every A-operator group in $\mathfrak{L}_A$, where A is a ring of operators, is an A-group;
every module over a ring A is an abelian A-group;
every free group with exponents over A in the sense of Lyndon is an A-group;
every nilpotent group with exponents over a binomial ring A which was introduced by Ph. Hall [4] is an A-group;
every pro-p group is a $\mathbb{Z}_p$-group over the ring $\mathbb{Z}_p$ of p-adic integers;
every profinite group is a $\widehat{\mathbb{Z}}$-group, where $\widehat{\mathbb{Z}}$ is the total completion of $\mathbb{Z}$ with respect to the profinite topology;
every complex (real) unipotent Lie group is a $\mathbb{C}$-group ($\mathbb{R}$-group).

Every group with exponents over A in the sense of Lyndon is a multioperator group (cf. [5]) in signature

$$\sigma_A = \langle \cdot, {}^{-1}, f_\alpha \mid \alpha \in A \rangle,$$

where f_α is the unary operation of raising to the power α; i.e., $f_\alpha(g) = g^\alpha$.

Since axioms (1)–(3) are identities and axiom (4) is a quasi-identity, the definition of group with exponents implies the following

Proposition 1. 1. *The class $\mathfrak{L}_A$ is a variety in signature σ_A.*
2. *The class $\mathfrak{M}_A$ is a quasivariety in signature σ_A.*

1.2. Faithfulness and torsion. Let G be an A-group. We introduce the following notation:

$$x^A = \{x^\alpha \mid \alpha \in A\},\ x \in G, \quad X^A = \bigcup_{x \in X} x^A,\ X \subseteq G.$$

DEFINITION 3. A nonzero element $\alpha \in A$ *acts faithfully* on G if $G^\alpha \neq 1$. The ring A of scalars *acts faithfully on G* (A is a *faithful ring of scalars* of G) if every nonzero element of A acts faithfully on G.

We put $\mathrm{Ann}_A G = \{\alpha \in A \mid G^\alpha = 1\}$.

Proposition 2. *The following assertions are valid for every A-group G:*
(1) *$\mathrm{Ann}_A G$ is a two-sided ideal of A;*
(2) *the action of A on G induces an action of the quotient ring $\overline{A} = A/\mathrm{Ann}_A G$ on G such that G becomes a faithful $\overline{A}$-group.*

Proposition 2 allows one to pass from an arbitrary action of A on G to a faithful action on G; however, the latter is now that of the ring $\overline{A}$.

DEFINITION 4. An element $g \in G$ is said to be a *torsion element* if $g^\alpha = 1$ for some $\alpha \in A$, $\alpha \neq 0$. The right ideal $\mathfrak{D}(g) = \{\alpha \in A \mid g^\alpha = 1\}$ is called the *exponent ideal* of g. A group G without nonidentical torsion elements is called an *A-torsion-free group*.

PROPERTY 3. *Every A-torsion-free group is a faithful A-group.*

PROPERTY 4. *Let g be a nonidentical element of G. If α is invertible in A then $g^\alpha \neq 1$.*
Properties 3 and 4 imply

Proposition 3. *If A is a skew field then every A-group is a faithful A-torsion-free group.*

Now we give an example of a faithful A-group each of whose elements is a torsion element.

EXAMPLE 2. Let $G = \bigoplus_{i=1}^{\infty} C(n)$, where $C(n)$ is a cyclic group of order n. Then G is a faithful torsion $\mathbb{Z}$-group.

1.3. Morphisms. In this subsection we introduce various types of mappings between groups with exponents.

DEFINITION 5. Let $G, H \in \mathfrak{L}_A$. A homomorphism $\varphi : G \to H$ is said to be

(a) a *linear homomorphism*, or an *A-homomorphism*, if $(g^\alpha)^\varphi = (g^\varphi)^\alpha$ for all $g \in G$ and $\alpha \in A$;

(b) a *semilinear homomorphism* if there exists an endomorphism θ of the ring A such that $(g^\alpha)^\varphi = (g^\varphi)^{\theta(\alpha)}$ for all $g \in G$ and $\alpha \in A$;

(c) a *geometric homomorphism* if $(g^A)^\varphi = (g^\varphi)^A$ for all $g \in G$.

In a natural way we define A-mono(-epi, -iso, and -auto)morphisms.

Let A act faithfully on G as well as on H.

QUESTION: Given a G-semilinear isomorphism $\varphi : G \to H$, is θ necessarily an automorphism of the ring A?

Since the composition of A-automorphisms is an A-automorphism too, the set of all A-automorphisms constitutes some group $\mathrm{Aut}_A(G)$. Unfortunately, $\mathrm{Aut}_A(G)$ is not necessarily an A-group.

EXAMPLE 3. Let G be a vector space of dimension n over the field $\mathbb{Q}$. Then $G \in \mathfrak{M}_\mathbb{Q}$ and $\mathrm{Aut}_\mathbb{Q} G \simeq GL_n(\mathbb{Q})$. However, the group $GL_n(\mathbb{Q})$ is not a group with unique roots. For instance, there is no square root for the matrix

$$\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}.$$

DEFINITION 6. A *projective transformation* $\overline{\varphi} : G \to H$ is defined to be an isomorphism $\theta : L(G) \to L(H)$ of the respective lattices $L(G)$ and $L(H)$ of A-subgroups of G and H.

1.4. *A*-subgroups and ideals. Let G be an A-group.

DEFINITION 7. A subgroup $H \le G$ is called an *A-subgroup* if $H^A = H$. A subgroup H is *A-generated* by a set $X \subset G$ if H is the smallest A-subgroup of G containing X. We use the notation $H = \langle X \rangle_A$.

Thus, A-subgroups of G are the subgroups of G regarded as a multioperator group.

PROPERTY 5. *Let $X \subseteq G$. Assign $X_0 = \langle X \rangle$ and $X_{n+1} = \langle X_n^A \rangle$, where $\langle X \rangle$ denotes the subgroup (not A-subgroup!) generated by X. We have*

$$\langle X \rangle_A = \bigcup_{n=0}^{\infty} X_n.$$

PROPERTY 6. *If $H \trianglelefteq G$ then $\langle H \rangle_A \trianglelefteq G$.*

□ By Property 5, $\langle H \rangle_A = \bigcup_{n=0}^{\infty} H_n$, where $H_{n+1} = \langle H_n^A \rangle$. By axiom (3), $x^{-1} H_{n+1} x = \langle (x^{-1} H_n x)^A \rangle = \langle H_n^A \rangle$ for every $x \in G$. The last equality is obtained by induction on n. It follows that

$$x^{-1} \langle H \rangle_A x = \bigcup_{n=0}^{\infty} x^{-1} H_n x = \bigcup_{n=0}^{\infty} H_n = \langle H \rangle_A. \ \square$$

Notice that, given an arbitrary normal A-subgroup H of G, the quotient group G/H does not necessarily enjoy some natural A-group structure. Below we define the notion of an ideal H of the group G that satisfies certain conditions which allow one to induce A-structure in G/H. Of course, the definitions of A-ideal differ in the categories $\mathfrak{L}_A$ and $\mathfrak{M}_A$.

DEFINITION 8. A normal A-subgroup H of a group $G \in \mathfrak{L}_A$ is called an $\mathfrak{L}_A$-*ideal* if $(gh)^\alpha \in g^\alpha H$ for all $g \in G$, $h \in H$, and $\alpha \in A$.

Proposition 4. *Given groups $G, H \in \mathfrak{L}_A$, the following assertions hold:*
(1) if $\varphi : G \to H$ is an A-homomorphism then $\ker(\varphi)$ is an $\mathfrak{L}_A$-ideal of G;
(2) if H is an $\mathfrak{L}_A$-ideal of G then the action of A on G induces an action of A on G/H by the rule $(gH)^\alpha = g^\alpha H$, $g \in G$ which makes G/H into an $\mathfrak{L}_A$-group.

To define an $\mathfrak{M}_A$-ideal, we need some preliminary notions.

DEFINITION 9. Given $g, h \in G$ and $\alpha \in A$, we call the element $(g,h)_\alpha = h^{-\alpha}g^{-\alpha}(gh)^\alpha$ the α-*commutator* of g and h.

It is clear that $(gh)^\alpha = g^\alpha h^\alpha (g,h)_\alpha$ and $G \in \mathfrak{M}_A \Leftrightarrow ([g,h]_\alpha = 1 \to (g,h)_\alpha = 1)$. The preceding equivalence leads to the following definition of $\mathfrak{M}_A$-ideal.

DEFINITION 10. A normal A-subgroup $H \trianglelefteq G$, $G \in \mathfrak{L}_A$, is said to be an $\mathfrak{M}_A$-*ideal* if

$$[g,h] \in H \Rightarrow (g,h)_\alpha \in H$$

for all $g, h \in G$ and $\alpha \in A$.

Proposition 5. *Let $G \in \mathfrak{L}_A$. Then*
(1) if H is an $\mathfrak{M}_A$-ideal of G then H is an $\mathfrak{L}_A$-ideal of G;
(2) if $\varphi : G \to H$ is an A-homomorphism of groups in $\mathfrak{M}_A$ then $\ker(\varphi)$ is an $\mathfrak{M}_A$-ideal of G;
(3) if H is an $\mathfrak{M}_A$-ideal of G then $G/H \in \mathfrak{M}_A$.

□ (1): Let H be an $\mathfrak{M}^A$-ideal of G. Then we have $(gh)^\alpha = g^\alpha h^\alpha (g,h)_\alpha$ for $g \in G$, $h \in H$, and $\alpha \in A$. Since H is a normal subgroup of G, we have $[g,h] \in H$; therefore, $(g,h)_\alpha \in H$. It follows that $h^\alpha (g,h)_\alpha \in H$ and $(g,h)_\alpha \in g^\alpha H$; i.e., H is an $\mathfrak{L}_A$-ideal of G.

(2): Let $\varphi : F \to H$ be an A-homomorphism of groups in $\mathfrak{M}^A$. Then $C = \ker(\varphi)$ is a normal A-subgroup of G. If $[g,h] \in G$ then $[g^\varphi, h^\varphi] = [g,h]^\varphi = 1$. From here we infer the equality $(g^\varphi, h^\varphi)_\alpha = 1$ in H. It follows that $(g,h)_\alpha^\varphi = (g^\varphi, h^\varphi)_\alpha = 1$; i.e., $(g,h)_\alpha \in C$. Thus, C is an $\mathfrak{M}_A$-ideal.

(3): If H is an $\mathfrak{M}_A$-ideal of G then, by (1), H is an $\mathfrak{L}_A$-ideal of G; therefore, $G/H \in \mathfrak{L}_A$. By the definition of $\mathfrak{M}_A$-ideal, we have $[g,h] \in H \Rightarrow \forall \alpha \in A\ ((g,h)_\alpha \in H)$, which is equivalent to validity of axiom (4) for G/H. □

1.5. Operations over groups with exponents. We will show that the classes $\mathfrak{L}_A$ and $\mathfrak{M}_A$ are closed under the taking of direct and Cartesian products, as well as direct and inverse limits.

Let $G_i \in \mathfrak{L}_A$, $i \in I$. By $\overline{\prod} G_i$ and $\prod G_i$ we shall mean the Cartesian direct products of the groups G_i. Let $g \in \overline{\prod} G_i$, $g = (\dots, g_i, \dots)$, $\alpha \in A$. We define the action of A on G componentwise:

$$g^\alpha = (\dots, g_i^\alpha, \dots).$$

It is straightforward that if all groups G_i satisfy one of the axioms (1)–(4), then both groups $\overline{\prod} G_i$ and $\prod G_i$ satisfy the same axiom. This proves the following

Proposition 6. *The classes $\mathfrak{L}_A$ and $\mathfrak{M}_A$ are closed under direct and Cartesian products.*

Considering only A-homomorphisms in the standard definitions of direct and inverse limits, we can easily prove the following

Proposition 7. *The classes $\mathfrak{L}_A$ and $\mathfrak{M}_A$ are closed under direct and inverse limits.*

In the category of abelian groups the operations of direct product, direct limit, and inverse limit enjoy universality properties [6]. The corresponding operations in the category of groups with exponents enjoy similar properties as well. Here we confine ourselves to stating the corresponding universality properties.

Proposition 8 (universality of direct products). *Let $\varphi_i : G_i \to H$ be A-homomorphisms and let $\chi_i : G_i \to H$ be embeddings such that*

$$[\varphi_i(G_i), \varphi_j(G_j)] = 1, \quad i \neq j,\ i, j \neq I.$$

Then there exists a unique A-homomorphism $\psi : \prod_{j \in I} G_j \to H$ (independent of i) such that every diagram of the form

$$\begin{array}{ccc} G_i & \xrightarrow{\chi_i} & \prod_{i\in I} G_i \\ {\scriptstyle \varphi_i}\downarrow & \swarrow {\scriptstyle \psi} & \\ H & & \end{array} \qquad (i \in I)$$

commutes.

Denote by $G_* = \varinjlim_{i\in I} G_i$ the limit group of a direct spectrum $\{G_i\ (i \in I); \pi_i^j\}$.

Proposition 9 (universality of direct limits). *Let π_i denote the projection of G_i into G_* and let $\sigma_i : G_i \to H$ be A-homomorphisms such that every diagram of the form*

$$\begin{array}{ccc} G_i & \xrightarrow{\pi_i^j} & G_j \\ {\scriptstyle \sigma_i}\downarrow & \swarrow {\scriptstyle \sigma_j} & \\ H & & \end{array} \qquad (i \leq j)$$

commutes. Then there exists a unique homomorphism $\sigma : G_* \to H$ such that every diagram of the form

$$\begin{array}{ccc} G_i & \xrightarrow{\pi_i} & G_* \\ {\scriptstyle \sigma_i}\downarrow & \swarrow {\scriptstyle \sigma} & \\ H & & \end{array} \qquad (i \in I)$$

commutes.

Denote by $G^* = \varprojlim_{i\in I} G_i$ the limit group of an inverse spectrum $\{G_i\ (i \in I); \pi_i^j\}$.

Proposition 10 (universality of inverse limits). *If H is an A-group and $\sigma_i : H \to G_i$ are A-homomorphisms such that every diagram of the form*

$$\begin{array}{ccc} G_i & \xrightarrow{\pi_i^j} & G_j \\ {\scriptstyle \sigma_i}\downarrow & \swarrow {\scriptstyle \sigma_j} & \\ H & & \end{array} \qquad (i \leq j)$$

commutes then there exists a unique homomorphism $\sigma : H \to G^*$ such that every diagram of the form

$$\begin{array}{ccc} G_i & \xrightarrow{\pi_i^j} & G^* \\ {\scriptstyle \sigma_i}\downarrow & \swarrow {\scriptstyle \sigma} & \\ H & & \end{array} \qquad (i \in I)$$

commutes.

§ 2. Tensor Completions

Now we study the key operation in the class of groups with exponents, that of tensor completion. It naturally generalizes the notion of scalar extension for modules to the noncommutative case. For the class of nilpotent groups, the idea of the generalization was exposed in the authors' article [7]. We will exploit the tensor completion while defining free constructions in the category of groups with exponents, the concept of free A-group inclusively.

DEFINITION 11. Let G be an A-group and let $\mu : A \to B$ be a homomorphism of rings. A B-group $G^{B,\mu}$ is said to be a *tensor B-completion* of G if it is universal in the following sense:

(1) there exists an A-homomorphism $\lambda : G \to G^{B,\mu}$ such that $\lambda(G)$ B-generates the group $G^{B,\mu}$; i.e., $\langle \lambda(G) \rangle_B = G^{B,\mu}$;

(2) for an arbitrary B-group H and an arbitrary A-homomorphism $\varphi : G \to H$ consistent with μ (i.e., such that $(g^\alpha)^\varphi = (g^\varphi)^{\mu(\alpha)}$), there exists a B-homomorphism $\psi : G^{B,\mu} \to H$ making the diagram

$$\begin{array}{ccc} G_i & \xrightarrow{\ \lambda\ } & G^{B,\mu} \\ {\scriptstyle\varphi}\big\downarrow & \swarrow{\scriptstyle\psi} & \\ H & & \end{array}$$

commutative.

If an A-group G is abelian then, by item (1) of Definition 11 and Properties 2 and 5, the group $G^{B,\mu}$ is abelian too; i.e., $G^{B,\mu}$ is a B-module. Furthermore, $G^{B,\mu}$ satisfies the universality property of the tensor product $G \otimes_A B$ of the A-module G by the ring B. Thus, $G^{B,\mu} \simeq G \otimes_A B$.

Now we demonstrate that the notion of tensor completion is a natural generalization of the notion of scalar extension for modules and that the new notion possesses most attributes of the former.

In what follows, we usually fix the ring homomorphism $\mu : A \to B$, so that we shall write G^B rather than $G^{B,\mu}$.

Theorem 1 (an existence theorem). *For every A-group G and every ring homomorphism $\mu : A \to B$ there exists a B-completion.*

□ We introduce an equivalence relation in the class of all A-homomorphisms $\varphi : G \to H$ consistent with μ and such that H is a B-group B-generated by $\varphi(G)$. Given two homomorphisms φ_1 and φ_2 of this form, we set $\varphi_1 \sim \varphi_2$ if and only if $\ker(\varphi_1) = \ker(\varphi_2)$ and there exists a B-isomorphism ψ between H_1 and H_2 such that $\psi\varphi_1(g) = \varphi_2(g)$ for all $g \in G$. The collection I of the equivalence classes is a nonempty set of cardinality bounded by $|G|$ and $|B|$. We fix a representative $\varphi_i : G \to H_i$ in each equivalence class $i \in I$.

Let $G_0 = \prod\limits_{i \in I} H_i$ be the Cartesian product of the groups. Then G_0 is a B-group and the map $\lambda : g \mapsto (\dots, \varphi_i(g), \dots)$ is an A-homomorphism from G into G_0. Let $G^B = \langle \lambda(G) \rangle_B$ be the B-group in G_0 generated by the set $\lambda(G)$. We will show that G^B is the sought tensor B-completion of G. Since our construction guarantees that G^B meets the first requirement of universality, we only have to verify that the second is fulfilled.

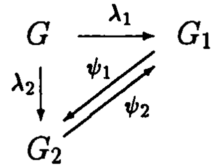
Let $\varphi : G \to H$ be an arbitrary A-homomorphism, consistent with μ, from G into a B-group H. We put $H_0 = \langle \lambda(G) \rangle_B \leq H$. For a suitable $i \in I$ we have $\varphi \sim \varphi_i$. Let an isomorphism ψ between H_i and H_0 make φ_i and φ equivalent. Consider the diagram

$$\begin{array}{ccccc} G & \xrightarrow{\ \lambda\ } & G^B & \xrightarrow{\ \epsilon\ } & G_0 \\ {\scriptstyle\varphi}\big\downarrow & & {\scriptstyle\pi}\big\downarrow & & {\scriptstyle\pi_i}\big\downarrow \\ H & \xleftarrow[\ \varepsilon\ ]{} & H_0 & \xleftarrow[\ \psi\ ]{} & H_i \end{array}$$

in which ϵ and ε denote inclusions, π_i denotes the canonical projection, and $\pi = \pi_{i|G^B}$. Then $\pi\varepsilon$ is the sought B-homomorphism consistent with μ and making the diagram commutative. □

Theorem 2 (uniqueness theorem). *For every A-group G and every ring homomorphism $\mu : A \to B$, the tensor B-completion of G is unique up to B-isomorphism.*

□ Let G_1 and G_2 be two arbitrary B-completions of G with respect to μ. By definition, some B-homomorphisms ψ_1 and ψ_2 make the diagrams



commutative. Put $f_1 = \psi_1\psi_2$ and $f_2 = \psi_2\psi_1$. Since f_i is identical on $\lambda_i(G)$ and $\lambda_i(G)$ B-generates G_i, we have $f_i = \mathrm{id}$; i.e., ψ_1 and ψ_2 are mutually inverse B-homomorphisms. □

In applications μ will mostly be an embedding of rings. However, even in this case, the homomorphism $\lambda : G \to G^{B,\mu}$ is not always an embedding. Since in the abelian case the group G^B results from by tensoring the A-module G by the ring B, appropriate examples can be found in many articles on commutative algebra and homology. The following proposition describes the situation in which λ is an embedding.

DEFINITION 12. We call an A-group G a *residually-B group with respect to a homomorphism μ* if for every $1 \neq g \in G$ there exists an A-homomorphism φ_g from G to a B-group H consistent with μ and such that $\varphi_g(g) \neq 1$.

Proposition 11. *Let an A-group G be a residually-B group with respect to a homomorphism μ. Then the homomorphism $\lambda : G \to G^B$ is an embedding.*

□ Let $1 \neq g \in G$ and let $\psi_g : G \to H$ be an A-homomorphism consistent with μ and such that $\varphi_g(g) \neq 1$. There exists a homomorphism $\psi : G^B \to H$ such that $\varphi_g = \psi\lambda$. Therefore, $\lambda(g) \neq 1$. □

§ 3. The Category of Groups with Exponents

We list basic categorical properties of tensor completion.

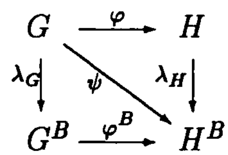
3.1. The category of A-groups. The following proposition is obvious.

Proposition 12. *The class $\mathfrak{M}_A$ of all A-groups is a category whose morphisms are all A-homomorphisms.*

The category of A-groups possesses practically all attributes of the category of groups.

From the categorical point of view, the above-introduced operation of tensor completion presents the *functor of tensor completion*. Let $\mu : A \to B$ be a homomorphism of rings. Basing on this homomorphism, we will construct a functor $\theta^{B,\mu}$ (writing θ^B from now on) which will relate the category $\mathfrak{M}_A$ of A-groups with the category $\mathfrak{M}_B$ of B-groups. The action of the map $\theta^B : \mathfrak{M}_A \to \mathfrak{M}_B$ on objects is determined by the rule $\theta^B(G) = G^B$, where G is an A-group and G^B is the tensor B-completion of G with respect to μ.

Define the action of θ^B on the morphisms of $\mathfrak{M}_A$. Let $H, G \in \mathfrak{M}_A$ and let $\varphi : G \to H$ be an A-homomorphism. Let $\lambda_G : G \to G^B$ and $\lambda_H : H \to H^B$ be the canonical maps of Section 2. Since the composition $\varphi \circ \lambda_H$ is an A-homomorphism from A into H^B consistent with μ, there exist B-homomorphisms ψ and ψ^B making the diagram



commutative. We assign $\theta^B(\varphi) = \varphi^B$.

Theorem 3. *If $\mu : A \to B$ is a ring homomorphism then θ^B is a covariant functor from the category $\mathfrak{M}_A$ to the category $\mathfrak{M}_B$.*

□ We will check one of the axioms of the definition of functor. Let $H = G$ and $\varphi = 1_G$. Verify that $(1_G)^B = 1_{G^B}$. Indeed, if $\lambda : G \to G^B$ is the canonical homomorphism then the restriction of $(1_G)^B$ to $\lambda(G)$ is identical. Since $G^B = \langle\lambda(G)\rangle_B$, the map $(1_G)^B$ is identical on the entire group G^B as well.

The remaining axioms of the definition of functor can be verified in a similar way. □

By a standard argument one can prove the following

Proposition 13. *For short exact sequences, the functor of tensor completion is right exact but not left exact in general.*

Let $A \xrightarrow{\mu_1} B \xrightarrow{\mu_2} C$ be a sequence of ring homomorphisms. Assign $\mu = \mu_1\mu_2 : A \to C$. These three homomorphisms determine a triple of functors

$$\Phi^{B,\mu_1} : \mathfrak{M}_A \to \mathfrak{M}_B, \quad \Phi^{C,\mu_2} : \mathfrak{M}_B \to \mathfrak{M}_C, \quad \Phi^{C,\mu} : \mathfrak{M}_A \to \mathfrak{M}_C.$$

We define the composition of functors Φ^{B,μ_1} and Φ^{C,μ_2} in a natural way.

Proposition 14. *With the above notation,*

$$\Phi^{B,\mu_1} \circ \Phi^{C,\mu_2} = \Phi^{C,\mu}.$$

Let $\mu : A \to B$ be an arbitrary ring homomorphism and let $\mathrm{Im}(\mu) = B_0$. Then μ is canonically factored into the product of an epimorphism $\mu_1 : A \to B_0$ and a monomorphism $\mu_2 : B_0 \to B$. By Proposition 14, we have $\Phi^{B,\mu_1} \circ \Phi^{C,\mu_2} = \Phi^{C,\mu}$. In this case we shall speak about the canonical factorization of the functor of tensor completion. By that reason, the proofs of the subsequent theorems related to the construction of tensor completion reduce in a natural way to examining the following two cases:

(a) μ is an epimorphism of rings;

(b) μ is an embedding of rings.

The operation of tensor completion commutes with the operations of direct product and direct limit, but, in general, does not commute with the operations of Cartesian product and inverse limit [8].

The fact that the tensor completion commutes with direct limits allows one to reduce many questions on completions to the case of finitely generated groups. Indeed, let $\{G_i \ (i \in I); \pi_i^j\}$ be a direct spectrum of G composed of finitely generated groups G_i. Then $G = \varinjlim_{i\in I} G_i$ and $G^B \simeq \varinjlim_{i\in I} G_i^B$.

3.2. Supercategories and subcategories of the groups with exponents. By technical reasons, it is convenient to introduce some supercategories and subcategories of the groups with exponents. For instance, eliminating axiom (4) in the definition of a group with exponents we obtain Lyndon's supercategory $\mathfrak{L}_A \supset \mathfrak{M}_A$. If, in addition, we remove axiom (3) then we obtain a wider supercategory $\mathfrak{R}_A \supset \mathfrak{L}_A$. On the other hand, one can revise the general notion of a group with exponents (for instance, in order to adapt it to a particular variety of groups) by adjoining some other axioms to axioms (1)–(4). In this way, there appears the category $\mathfrak{H}_A$ of nilpotent A-groups in the sense of Ph. Hall [4].

The following question arises: which new categories of groups with exponents enjoy properties similar to those of 3.1?

Examination of the arguments of the previous subsection shows that, for tensor completion in a given class $\mathfrak{R} \subset \mathfrak{R}_A$ of groups (in signature σ_A of A-operator groups) to exist, it is sufficient for the class $\mathfrak{R}$ to be closed under Cartesian products and subsystems; moreover, $\mathfrak{R}$ must contain the trivial group. The conditions indicated are automatically satisfied if $\mathfrak{R}$ is a quasivariety of groups in signature σ_A. Thus, removing some axioms from (1)–(4) (for instance, by passing to the supercategories $\mathfrak{L}_A$ or $\mathfrak{R}_A$) or adding quasi-identities as new axioms (for instance, by considering the subcategory $\mathfrak{H}_{A,n}$ of nilpotent A-groups in the sense of Ph. Hall of nilpotency class n) we obtain various categories of groups with exponents in which tensor completion exists and is unique up to A-isomorphism; the operation of tensor completion is a functor, and this functor admits canonical factorization.

3.3. The category of partial groups with exponents. It is convenient to construct a tensor completion of a given group step by step by successively "adjoining exponents." This method leads to the notion of partial A-group. Also, some group operations over A-groups lead to partial A-groups. Let A be a ring and let G be a group.

DEFINITION 13. The group G is said to be a *partial A-group* if raising to a power is defined for some (not necessarily all) pairs (g, α), with each of the axioms (1)–(4) of the definition of A-group fulfilled whenever both sides of the axiom are defined. We denote the class of partial A-groups by $\mathfrak{P}_A$.

EXAMPLE 4. Let A be a subring of a ring B. Then every A-group is a partial B-group.

DEFINITION 14. Let $H, G \in \mathfrak{P}_A$. A homomorphism of groups $\varphi : G \to H$ is called a *partial A-homomorphism* if $(g^\alpha)^\varphi = (g^\varphi)^\alpha$ for all pairs (g, α) such that the element g^α is defined.

It is straightforward that $\mathfrak{P}_A$ is a subcategory of $\mathfrak{M}_A$.

Proposition 15. *In the category $\mathfrak{P}_A$ there exist direct, Cartesian, and free products as well as direct and inverse limits.*

□ We examine the case of free products as an example. Let $G_1, G_2 \in \mathfrak{P}_A$ and $G = G_1 * G_2$. If H is a subgroup of G conjugate with one of the factors then we define a partial action of A on H so as to satisfy axiom (3) of the definition of a group with exponents. If an element g is not conjugate with any element of the factors and $\alpha \notin \mathbb{Z}$ then we leave g^α undefined. With this definition of the action of A on G, the free product becomes a partial A-group. □

The abelian partial A-groups in the category $\mathfrak{P}_A$ are partial A-modules. Many general notions for A-modules can be carried over into the category of partial A-modules.

For our purposes, it is important that the notion of tensor product of A-modules can be generalized to the notion of tensor product of partial A-modules. We define this notion in line with [9].

Proposition 16. *Let M and N be partial A-modules. There exists a pair (T, g) composed of a partial A-module T and a partial A-bilinear map $g : M \times N \to T$ with the following property:*

for every partial A-module P and every A-bilinear map $f : M \times N \to P$ there exists a unique A-linear map $f' : T \to P$ such that $f = gf'$. (In other words, every partial bilinear map can be factored through T.)

If (T, g) and (T', g') are two pairs with this property then there exists an A-isomorphism $j : T \to T'$ for which $gj = g'$.

□ Uniqueness is proved in a standard fashion. We will prove existence. Consider a free A-module C. The elements of C are formal linear combinations of elements of $M \times N$ with coefficients in A, i.e. expressions of the form

$$\sum_{i=1}^{n} \alpha_i(x_i, y_i), \quad \text{where} \quad \alpha_i \in A,\ x_i \in M,\ y_i \in N.$$

Let D denote the submodule of C that is generated by the elements of the form

$$(x + x', y) - (x, y) - (x', y), \quad (x, y + y') - (x, y) - (x, y'),$$
$$(\alpha x, y) - (\alpha x, y), \quad (x, \alpha y) - (\alpha x, y)$$

on condition that αx is defined in M and αy in N.

Put $T = C/D$. Given a basis element $(x, y) \in C$, denote its image in T by $x \otimes y$. The module T is generated by the elements of the form $x \otimes y$; and it is clear from the definition that they satisfy the relations

$$(x + x') \otimes y = x \otimes y + x' \otimes y,\ x \otimes (y + y') = x \otimes y + x \otimes y',\ (\alpha x) \otimes y = \alpha(x \otimes y) = x \otimes (\alpha y).$$

In other words, the map $g : M \times N \to T$ defined by the rule $g(x, y) = x \otimes y$ is A-bilinear.

Any map f from the product $M \times N$ into a partial A-module P is extendable by linearity to an A-module homomorphism $\bar{f} : C \to P$. If, in addition, f is partially A-bilinear then f vanishes on the generators of D and hence on the entire D. Therefore, f induces a unique partial A-homomorphism f' from the module T into P such that $f'(x \otimes y) = f(x, y)$. Thus, the pair (T, g) enjoys the desired property. □

REMARK 1. The above constructed module T is called the *tensor product* of modules M and N and is denoted by $M \otimes_A N$, or simply $M \otimes N$ in case it is clear which ring A is implied. Notice that the tensor product $M \otimes N$ of partial A-modules is an ordinary A-module.

REMARK 2. If, in the definition of tensor completion, we replace the condition that $\lambda : G \to G^B$ and $\varphi : G \to H$ are A-homomorphisms with the condition that they are partial A-homomorphisms then we arrive at the definition of tensor completion of a partial A-group. In exactly the same way one can define the categories of partial A-groups $\mathfrak{P}_{\mathfrak{L}_A}$, $\mathfrak{P}_{\mathfrak{N}_A}$, and $\mathfrak{P}_{\mathfrak{H}_A}$ that correspond to $\mathfrak{L}_A$, $\mathfrak{N}_A$, and $\mathfrak{H}_A$.

§ 4. Free Constructions

We introduce the notion of a free A-group. Let A be a commutative ring with unity and let X be an arbitrary set.

DEFINITION 15. An A-group $F_A(X)$ with an A-generating set X is called a *free A-group with basis X* if, given an arbitrary A-group G, every map $\varphi_0 : X \to G$ is extendable to a homomorphism $\varphi : F_A(X) \to G$. The set X is called the *set of free A-generators* for $F_A(X)$. The cardinal number $|X|$ is called the *rank* of $F_A(X)$.

Theorem 4. *For any X and A, the free A-group $F_A(X)$ exists and is unique up to A-isomorphism.*

□ Let $F(X)$ be an ordinary free group. Then a tensor A-completion of it is a free A-group with basis X. Indeed, let $\varphi_0 : X \to G$ be an arbitrary map from X into an A-group G:

$$\begin{array}{ccc} X & \longrightarrow & F(X) \\ {\scriptstyle \varphi_0}\downarrow & \swarrow{\scriptstyle \varphi_1} & \downarrow \\ G & \xleftarrow{\ \varphi\ } & (F(X))^A \end{array} .$$

The map φ_0 is extendable to a homomorphism $\varphi_1 : F(X) \to G$ by the property of a free group. This homomorphism is, in turn, extendable to an A-homomorphism $\varphi : (F(X))^A \to G$. It follows that $(F(X))^A$ is a free A-group with basis X. Uniqueness follows from that of tensor completion. □

Now we introduce the construction of free product in the category of A-groups.

DEFINITION 16. Let G_i, $i \in I$, be A-groups. An A-group $*_A G_i$ is called the *free product in the category* $\mathfrak{M}_A$ if the A-homomorphisms $\varphi_i : G_i \to *_A G_i$ are such that for arbitrary A-homomorphisms $\psi_i : G_i \to H$, where H is an arbitrary A-group, there exists an A-homomorphism $\psi : *_A G_i \to H$ making the diagrams

$$\begin{array}{ccc} X & \longrightarrow & *_A G_i \\ {\scriptstyle \varphi_i}\downarrow & \swarrow{\scriptstyle \psi} & \\ H & & \end{array} \qquad (i \in I)$$

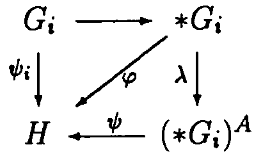
commutative and $*_A G_i$ is A-generated by the set $\{\varphi_i(g_i),\ g_i \in G_i,\ i \in I\}$.

By the categorical argument, the group $*_A G_i$ is unique up to A-isomorphism.

Theorem 5. *Let A be a ring containing the ring $\mathbb{Z}$ of integers as a subring and let G_i, $i \in I$, be some set of A-groups. Then $*_A G_i \simeq (*G_i)^A$.*

□ Let $\varphi_i^0 : G_i \to *G_i$ be canonical embeddings. Since, by Proposition 15, the free product possesses the structure of a partial A-group, the construction of tensor completion applies to it.

Let $\lambda : *G_i \to (*G_i)^A$ be the canonical map from the definition of tensor completion. Put $\lambda \circ \varphi_i^0 = \varphi_i$. The family $\varphi_i : G_i \to (*_A G_i)^A$ is a family of A-homomorphisms. Let $\psi_i : G_i \to H$ be arbitrary A-homomorphisms. In order to prove that $(*G_i)^A$ is a free product in the category $\mathfrak{M}_A$, we must complete the diagram



up to a commutative one.

By the definition of free product in the category $\mathfrak{M}_A$, there exists a partial A-homomorphism $\varphi : *_A G_i \to H$. By universality of tensor completion, there exists an A-homomorphism ψ extending φ. It is the one sought. The condition that $(*_A G_i)^A$ is generated by the images $\varphi_i(G_i)$ is satisfied too. Therefore, $(*_A G_i)^A$ is a free product in $\mathfrak{M}_A$. □

It is easy to give a categorical definition of the construction of amalgamated free product for A-groups. However, it is not always possible to define some structure of a partial A-group on such a product. Therefore, one cannot apply the construction of tensor A-completion to it. In case some structure of a partial A-group is definable, the explicit description for the construction of amalgamated free product is the same as that for free products.

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