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Algebraic geometry over groups III: Elements of model theory

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Abstract

One of the main results of this paper is that elementary theories of coordinate groups $\Gamma(Y_i)$ of irreducible components Y_i of an algebraic set Y over a group G are interpretable in the coordinate group $\Gamma(Y)$ of Y for a wide class of groups G. This implies, in particular, that one can study model theory of $\Gamma(Y)$ via the irreducible coordinate groups $\Gamma(Y_i)$. This result is based on the technique of orthogonal systems of subdirect products of domains, which we develop here. It has some other interesting applications, for example, if H is a finitely generated group from the quasi-variety generated by a free non-abelian group F, then H is universally equivalent either to a unique direct product F^l of l copies of F or to the group $F^l \times Z$, where Z is an infinite cyclic.

Keywords: Algebraic geometry; Groups; Model theory; Quasi-varieties; Irreducible components; Definability

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1. Introduction

1.1. Some notions from model theory

It has been shown in [9] that basic notions of algebraic geometry over groups have interesting connections with logic and universal algebra. We recall here a few necessary definitions and refer to [9] for details.

The standard language of group theory, which we denote by L, consists of a symbol for multiplication $\cdot$, a symbol for inversion $^{-1}$, and a symbol for the identity 1.

Let $X = \{x_1, \dots, x_n\}$ be a finite set of variables, $X^{-1} = \{x^{-1} \mid x \in X\}$, and $X^{\pm 1} = X \cup X^{-1}$. A *group word* in variables X is a word $S(X)$ in the alphabet $X^{\pm 1}$. Observe, that every term in the language L is logically equivalent (modulo the axioms of group theory) to a group word in X. An *atomic formula* in the language L is a formula of the type $S(X) = 1$. Sometimes we refer to atomic formulas in L as (coefficient-free) equations, and vice versa. A *Boolean combination* of atomic formulas in the language L is a disjunction of conjunctions of atomic formulas or their negations. It follows from general results on disjunctive normal forms that every formula $\Phi(X)$ in L is logically equivalent to a formula of the type

$$Q_1 z_1 Q_2 z_2 \dots Q_n z_m \Psi(X, Z),$$

where $Q_i \in \{\forall, \exists\}$, $Z = \{z_1, \dots, z_m\}$, and $\Psi(X, Z)$ is a Boolean combination of atomic formulas in variables $X \cup Z$. If in the formula $\Phi(X)$ the set of free variables X is empty then Φ is called a *sentence* in L. In the sequel we assume that all formulas are in L (if not said otherwise) and omit mentioning L.

If $\Phi(X)$ is a formula and G is a group, then for an n-tuple of elements $g = (g_1, \dots, g_n)$ from G we write $G \models \Phi(g)$ if $\Phi(X)$ holds in G on elements $(g_1, \dots, g_n)$. By $\Phi(G)$ we denote the truth set of Φ:

$$\Phi(G) = \bigl\{ g \in G^n \bigm| G \models \Phi(g) \bigr\}.$$

If G is a group, then the set $\mathrm{Th}(G)$ of all sentences which are valid in G is called the *elementary theory* of G. Two groups G and H are *elementarily equivalent* if $\mathrm{Th}(G) = \mathrm{Th}(H)$. The theory $\mathrm{Th}(G)$ is *decidable* if there is an algorithm which for every sentence ϕ determines whether or not ϕ is true in G.

A class of groups $\mathcal{K}$ is *axiomatic* if there exists a set of sentences Σ such that $\mathcal{K}$ consists precisely of all groups satisfying all formulas from Σ. In this event we say that Σ is a set of *axioms* for $\mathcal{K}$. For a class of groups $\mathcal{K}$ denote by $\mathrm{Th}(\mathcal{K})$ the elementary theory of $\mathcal{K}$, i.e., the set of all sentences of which are true in every group from $\mathcal{K}$. If $\mathcal{K} = \{H\}$ then we write $\mathrm{Th}(H)$ instead of $\mathrm{Th}(\{H\})$ and use this approach in all similar circumstances.

The notion of *interpretation* provides one of the most powerful tools in modern model theory (see, for example, [5,10,11]). It can be defined for arbitrary algebraic structures, but we restrict ourselves to groups only.

A *group code* C is a set of formulas

$$C = \bigl\{ U(X, P), E(X, Y, P), \mathrm{Mult}(X, Y, Z, P), \mathrm{Inv}(X, Y, P) \bigr\}, \tag{1}$$

where X, Y, Z, P are tuples of variables with $|X| = |Y| = |Z|$. If $P = \emptyset$, then C is called an *absolute code* or 0-code.

Let C be a group code, H be a group, and B be an $|P|$-tuple of elements in H. We say that C (with parameters B) *interprets* a group $C(H, B)$ in H if the following conditions hold:

(1) the truth set $U(H, B)$ in H of the formula $U(X, B)$ (with parameters B) is non-empty;
(2) the truth set of the formula $E(X, Y, B)$ (with parameters B) defines an equivalence relation $\sim_B$ on $U(H, B)$;
(3) the formulas $\mathrm{Mult}(X, Y, Z, B)$ and $\mathrm{Inv}(X, Y, B)$ define, correspondingly, a binary operation ($Z = Z(X, Y)$) and a unary operation ($Y = Y(X)$) on the set $U(H, B)$ compatible with the equivalence relation $\sim_B$;
(4) the set of equivalence classes $U(H, B)/\sim_B$ forms a group with respect to the operations defined by $\mathrm{Mult}(X, Y, Z, B)$ and $\mathrm{Inv}(X, Y, B)$. We denote this group by $C(H, B)$.

We say that a group G is *interpretable* (or *definable*) in a group H if there exists a group code C and a set of parameters $B \subset H$ such that $G \simeq C(H, B)$. If C is 0-code then G is *absolutely* or 0-*interpretable* in H. The following two types of interpretations are crucial. Let G be a definable subgroup of a group H, i.e., there exists a formula $U(x, P)$ and a set of parameters $B \subset H$ such that

$$G = \{g \in H \mid H \models U(g, B)\}.$$

Then G is interpretable in H by the code

$$C_G = \{U(x, P),\ x = y,\ xy = z,\ y = x^{-1}\}$$

with parameters B. If in addition G is a normal subgroup of H, then the code

$$C_{H/G} = \{x = x,\ \exists v\big(x = yv \wedge U(v, P)\big),\ z = xy,\ y = x^{-1}\}$$

interprets the factor-group H/G in H with parameters B.

Every group code (1) determines a *translation* T_C which is a map from the set of all formulas $\mathcal{F}_L$ in the language L into itself. We define T_C by induction as follows:

(1) $T_C(x = y) = E(X, Y, P)$;
(2) $T_C(xy = z) = \mathrm{Mult}(X, Y, Z, P)$ and $T_C(x^{-1} = y) = \mathrm{Inv}(X, Y, P)$;
(3) if $\phi, \psi \in \mathcal{F}_L$ and $\circ \in \{\wedge, \vee, \rightarrow\}$, then

$$T_C(\phi \circ \psi) = T_C(\phi) \circ T_c(\psi) \quad \text{and} \quad T_C(\neg\phi) = \neg T_C(\phi);$$

(4) if $\phi \in \mathcal{F}_L$, then

$$T_C\big(\exists x \phi(x)\big) = \exists X\big(U(X, P) \wedge T_C(\phi)\big),$$

$$T_C\big(\forall x \phi(x)\big) = \forall X\big(U(X, P) \to T_C(\phi)\big).$$

Observe, that the formula $T_C(\phi)$ can be constructed effectively from ϕ.

Now we are ready to formulate the fundamental (but easy to prove) property of interpretations.

Let a group code C interprets (with parameters B) a group G in a group H, and let $\lambda : G \to C(H, B)$ be the corresponding isomorphism. Then for every formula $\phi(X)$ and every $|X|$-tuple A of elements from G the following equivalence holds:

$$G \models \phi(A) \quad \Longleftrightarrow \quad H \models T_C(\phi)\big(A^\lambda, B\big).$$

In particular, a sentence ϕ holds in G if and only if $T_C(\phi)(B)$ holds in H.

If C is a 0-code, then $C(H)$ inherits some model theoretic properties of H. For example, if the theory $\mathrm{Th}(H)$ is decidable, or λ-stable, or has finite Morley rank, then so is the theory $\mathrm{Th}(C(H))$ (it follows directly from the fundamental property of translations). Moreover, if $H \equiv K$ then $C(H) \equiv C(K)$.

Sometimes, we cannot 0-interpret a group G in a group H. In this case, however, one can try to 0-interpret the elementary theory $\mathrm{Th}(G)$ in H. To explain, we need the following definition. Let G and H be groups. We say that the elementary theory $\mathrm{Th}(G)$ of G is interpretable in the group H if there exists a group code C of the type (1) and a formula $\Psi(P)$ such that $\mathrm{Th}(G) = \mathrm{Th}(C(H, B))$ for any set of parameters $B \subset H$ that satisfies the formula $\Psi(P)$ in H. It is not hard to see that the group G still satisfies the same model-theoretic properties as H (in the sense mentioned above). We refer to [6] and [7] for details.

One of the main results of this paper is that elementary theories of coordinate groups of irreducible components of an algebraic set Y over a group G are interpretable in the coordinate group of Y for a wide class of groups G. We will say more about it in the sequel.

1.2. *Direct products of domains and orthogonal systems*

In Section 2 we develop an approach to direct products of domains via orthogonal systems (of idempotents) similar to the classical one in the ring theory. To this end, following [1] we introduce a special binary operation, the so-called $\diamond$-product, on a group.

Let G be a group. For $x, y \in G$ put

$$x \diamond y = \big[\mathrm{gp}_G(x), \mathrm{gp}_G(y)\big].$$

We call a non-trivial element $x \in G$ a *zero-divisor* in G if there exists a non-trivial element $y \in G$ such that $x \diamond y = 1$. In this event we also say that y is *orthogonal* to x, and write $x \perp y$. A group G is termed a *domain* if it has no zero-divisors. The class of domains is fairly extensive, for example, it contains all non-abelian CSA groups and, in particular, all torsion-free hyperbolic groups. We refer to [1] for more details on zero-divisors and domains.

For a subset $S \subset G$ put

$$S^{\perp} = \{g \in G \mid \forall s \in S (g \diamond s = 1)\}.$$

It is easy to see that $S^{\perp}$ is a normal subgroup of G, it is called the *orthogonal complement* of S. In Section 2 we discuss various properties of $S^{\perp}$.

A system $E = \{e_1, \ldots, e_m\} \subset G$ is termed *orthogonal* if $e_i \neq 1$ and $e_i \diamond e_j = 1$ for all $1 \leqslant i \neq j \leqslant n$. In Proposition 1 we prove the following basic result:

Let $G = G_1 \times \cdots \times G_n$ be a finite direct product of domains $G_1, \ldots, G_n$. Then G has a unique (*up to a permutation of factors*) *finite direct decomposition into indecomposable groups. Moreover, it can be written as*

$$G = (e_1^{\perp})^{\perp} \times \cdots \times (e_n^{\perp})^{\perp},$$

where $\{e_1, \ldots, e_n\}$ is an arbitrary orthogonal system of n elements in G.

The unique factors $G_1, \ldots, G_n$ of the group G above are called *components* of G. It turns out that the elementary theory of G is completely determined by elementary theories of its components, which allows one to reduce model-theoretic problems about G to the corresponding problems for the components of G. This result is based on the following theorem.

Theorem A. *Let G be a finite direct product of domains. Then for each component G_i of G its elementary theory* $\mathrm{Th}(G_i)$ *is interpretable in the group G.*

Corollary A. *Let G be a finite direct product of domains $G_1, \ldots, G_n$. Then the following hold*:

(1) *If $G \equiv H$ then H is also a finite direct product of domains and if*

$$G = G_1 \times \cdots \times G_k, \qquad H = H_1 \times \cdots \times H_m$$

are their component decompositions, then $k = m$ and $G_i \equiv H_i$ (*after suitable ordering of factors*);

(2) $\mathrm{Th}(G)$ *is decidable if and only if* $\mathrm{Th}(G_i)$ *is decidable for every $i = 1, \ldots, k$*;

(3) $\mathrm{Th}(G)$ *is λ-stable* (*has finite Morley rank*) *if and only if* $\mathrm{Th}(G_i)$ *is λ-stable* (*has finite Morley rank*) *for every $i = 1, \ldots, k$.*

1.3. Subdirect products of domains

In Section 3 we generalize results on direct products of domains to subdirect products of domains.

Let $G = G_1 \times \cdots \times G_k$ be a direct product of groups G_i. A subgroup H of G is called a *subdirect product* of groups G_i if $\pi_i(H) = G_i$ for every $i = 1, \ldots, n$, where $\pi_i : G \to G_i$ is the canonical projection. An embedding

$$\lambda : H \hookrightarrow G_1 \times \cdots \times G_k \tag{2}$$

is called a *subdirect decomposition* of H if $\lambda(H)$ is a subdirect product of the groups G_i. Sometimes, we identify H with $\lambda(H)$ via λ. The subdirect decomposition (2) is termed *minimal* if $H \cap G_i \neq \{1\}$ for every $i = 1, \ldots, n$ (here G_i is viewed as a subgroup of G under the canonical embedding). It is easy to see that given a subdirect decomposition of H one can obtain a minimal one (by deleting non-essential factors).

In Proposition 3 we prove that a minimal subdirect decomposition of a group H into products of domains G_i is unique. We refer to the domains G_i as to *components* of H.

Theorem B. *Let H be a minimal subdirect product of domains. Then the elementary theory of each component of H is interpretable in the group H.*

This result allows one to study model theory of H via the components of H.

1.4. Algebraic geometry over groups

Section 4 contains some applications of the developed techniques to algebraic geometry over groups. To explain this we recall some basic definitions from [1].

Let $X = \{x_1, \ldots, x_n\}$ be a finite set. For a group G denote by $G[X]$ the free product $G * F(X)$ of G and a free group $F(X)$ with basis X. An element $f \in G[X]$ may be viewed as a word in the variables $X^{\pm 1}$ with coefficients in G. Given $p = (g_1, \ldots, g_n) \in G^n$, we can substitute $g_i^{\pm 1}$ for $x_i^{\pm 1}$ in f to obtain an element $f(p) \in G$. If $f(p) = 1$, we think of p as a solution of the equation $f = 1$. More generally, a subset S of $G[X]$ gives rise to a system of equations $S(X) = 1$ over G. The set

$$V_G(S) = \{ p \in G^n \mid f(p) = 1 \text{ for all } f \in S \}$$

is termed the *algebraic set* over G defined by S. Put

$$\operatorname{Rad}(S) = \{ f \in G[X] \mid f(p) = 1 \text{ for all } p \in Y \}.$$

Clearly, $\operatorname{Rad}(S)$ is a normal subgroup of $G[X]$, it is called the *radical* of S. The factor group $\Gamma(Y) = G[X]/\operatorname{Rad}(S)$ is termed the *coordinate group* of the algebraic set $Y = V_G(S)$.

One can define a so-called *Zariski topology* on G^n by taking algebraic sets as a sub-basis for closed sets. A group G is said to be *equationally Noetherian* if, for every $n > 0$ and any subset S of $G[x_1, \ldots x_n]$, there exists a finite subset S_0 of S such that $V_G(S) = V_G(S_0)$. Observe, that every linear group is equationally Noetherian, in particular, every free group is equationally Noetherian (see [1,2,4]). It turns out that a group G is equationally Noetherian if and only if the Zariski topology on G^n is Noetherian for every positive n. We recall that a topological space is *Noetherian* if it satisfies the descending chain condition on closed subsets. Noetherian topological spaces have a very nice property: every closed set is a finite union of irreducible ones (a non-empty subset Y is *irreducible* if it is not a union $Y = Y_1 \cup Y_2$ of proper subsets, each of which is relatively closed in Y). The following two results path the way for applications of the orthogonal systems into algebraic geometry over groups.

Theorem 1 [1]. *Every algebraic set Y over an equationally Noetherian group G is a finite union of irreducible algebraic sets, each of which is uniquely determined by Y.* (*They are called the irreducible components of V.*)

Theorem 2 [1]. *Let G be an equationally Noetherian group and Y be an algebraic set over G. If $Y_1, \ldots, Y_k$ are the irreducible components of Y then the coordinate group $\Gamma(Y)$ is a minimal subdirect product of the coordinate groups $\Gamma(Y_1), \ldots, \Gamma(Y_k)$.*

It might happen, in general, that the coordinate groups $\Gamma(Y_i)$ are not domains. So, to be able to apply our technique we need to put some restrictions on the group $G = \Gamma(Y)$. Recall [8] that a group G is called CSA if all maximal abelian subgroups of G are *malnormal* (a subgroup $M \leqslant G$ is malnormal if for every non-trivial $m \in M$ and $x \in G - M$ the conjugate $x^{-1}mx$ is not in M). We refer to [8] and [1] for a detailed discussion of CSA groups. Here we just observe that the class of CSA groups is quite wide (it contains, for example, all torsion-free hyperbolic groups) and that every non-abelian CSA group is a domain. Now, combining Theorems 2 and B, we obtain the following remarkable result.

Theorem C. *Let G be an equationally Noetherian, non-abelian CSA-group, Y be an algebraic set over G, and $\Gamma(Y)$ be the coordinate group of Y. Then for each component Y_i the elementary theory* $\mathrm{Th}(\Gamma(Y_i))$ *is interpretable in the group $\Gamma(Y)$.*

As we have seen above, this implies various model-theoretic results relating coordinate groups and their irreducible components.

1.5. Universal classes and axioms

In Section 5 we give another application of orthogonal systems to universal algebra. We begin with a few necessary definitions and refer to [9] for details.

A *universal sentence* in the language L is a formula of the type $\forall X \Phi(X, Y)$, where X and Y are tuples of variables, and $\Phi(X, Y)$ is a Boolean combination of atomic formulas in L.

A class of groups $\mathcal{K}$ is called *universal* if it can be axiomatized by a set of universal sentences. For a class of groups $\mathcal{K}$ denote by $\mathrm{Th}_\forall(\mathcal{K})$ the universal theory of $\mathcal{K}$, i.e., the set of all universal sentences of L which are true in every group from $\mathcal{K}$. Two groups H and K are *universally equivalent* (in writing $H \equiv_\forall K$) if $\mathrm{Th}_\forall(H) = \mathrm{Th}_\forall(K)$. The *universal closure* of $\mathcal{K}$ is the axiomatic class $\mathrm{ucl}(\mathcal{K})$ with the set of axioms $\mathrm{Th}_\forall(\mathcal{K})$.

Some universal classes are of particular interest. For example, a *variety* is a universal class axiomatized by a set of *identities*, i.e., universal formulas of the type

$$\forall X\left(\bigwedge_{i=1}^{m} r_i(X)=1\right), \tag{3}$$

where $r_i(X)$ is a group word in X. A class of groups $\mathcal{K}$ is called *a quasivariety* if it can be axiomatized by a set of quasi identities, which are universal formulas of the type

$$\forall X\left(\bigwedge_{i=1}^{m} r_i(X)=1 \rightarrow s(X)=1\right), \tag{4}$$

where $r_i(X)$ and $s(X)$ are group words in X.

For a class of groups $\mathcal{K}$ denote by $Q(\mathcal{K})$ the set of all quasi identities in the language L which hold in all groups from $\mathcal{K}$. Clearly, $Q(\mathcal{K})$ is a set of axioms of the *minimal* quasivariety $\mathrm{qvar}(\mathcal{K})$ containing $\mathcal{K}$. Observe, that every variety is a quasivariety.

A class $\mathcal{K}$ is called *a prevariety* if it is closed under taking subgroups and cartesian products. It is not hard to see that the minimal prevariety $\mathrm{pvar}(\mathcal{K})$ containing $\mathcal{K}$ consists of subgroups of cartesian products of groups from $\mathcal{K}$. It follows that for any class $\mathcal{K}$

$$\mathrm{pvar}(\mathcal{K}) \subseteq \mathrm{qvar}(\mathcal{K}) \subseteq \mathrm{var}(\mathcal{K}).$$

The following result links algebraic geometry over groups to universal algebra.

Theorem 3 [9]. *Let H be an equationally Noetherian group. Then the following hold*:

(1) *a finitely generated group K is the coordinate group of an algebraic set over H if and only if it belongs to* $\mathrm{qvar}(H)$;
(2) *a finitely generated group K, containing H as a subgroup, is the coordinate group of an irreducible algebraic set over H if and only if* $\mathrm{ucl}(K)=\mathrm{ucl}(H)$, *i.e., $K \equiv_\forall H$. In this event, K is also equationally Noetherian.*

The main result of Section 5 gives a description of the universal closure $\mathrm{ucl}(H)$ of any finitely generated group H from $\mathrm{qvar}(F)$, where F is a free non-abelian group. It turns out that each such class $\mathrm{ucl}(H)$ contains a unique representative. Namely, for a non-negative integer l define

$$G_{l,0}=\underbrace{F\times\cdots\times F}_{l}, \qquad G_{l,1}=\underbrace{F\times\cdots\times F}_{l}\times \mathbf{Z}.$$

Then the following result holds.

Theorem D. *Let F be a free non-abelian group and H be a finitely generated group from* $\mathrm{qvar}(F)$. *Then* $\mathrm{ucl}(H)=\mathrm{ucl}(G_{l,i})$ *for a suitable l and i. Moreover,* $\mathrm{ucl}(G_{l,i})=\mathrm{ucl}(G_{k,j})$ *if and only if $l=k$ and $i=j$.*

2. Orthogonal systems and direct products

Let G be a group. In Section 1.2 for any elements $x, y \in G$ we introduced the $\diamond$-product $x \diamond y$ and said that x is orthogonal to y ($x \perp y$) if $x \diamond y = 1$. In this section we use these notions to study direct decompositions of groups.

Recall that the *orthogonal complement* (or the $\diamond$-*annihilator*) of a subset $S \subseteq G$ is defined by

$$S^{\perp} = \{y \in G \mid \text{for all } x \in S \ x \perp y\}. \tag{5}$$

Sometimes, following ring theory, we denote $S^{\perp}$ by $\mathrm{Ann}(S)$. Notice that for any $S \subset G$

$$S^{\perp} = \mathrm{gp}_G(S)^{\perp}.$$

Lemma 1. *For any $S \subset G$ the orthogonal complement $S^{\perp}$ is a normal subgroup of G.*

Proof. Clearly

$$S^{\perp} = \bigcap \bigl\{ C\bigl(s^g\bigr) \bigm| g \in G,\ s \in S \bigr\},$$

hence it is normal, as required. $\square$

Note that G is a domain if and only if for any non-trivial $x \in G$, $x^{\perp} = \{1\}$.

Observe also, that for any x

$$x \perp x \quad \Longleftrightarrow \quad x \diamond x = 1 \quad \Longleftrightarrow \quad \mathrm{gp}_G(x) \quad \text{is abelian.}$$

More generally, an element $x \in G$ is $\diamond$-*nilpotent* of degree k if k is the minimal positive integer such that

$$\underbrace{\bigl(\ldots(x \diamond x) \diamond \ldots\bigr) \diamond x}_{k} = 1,$$

i.e., if $\mathrm{gp}_G(x)$ is a normal nilpotent subgroup of G of class k (see [1] for details). In this event $y \perp y$ for any central non-trivial y in $\mathrm{gp}_G(x)$. This argument suggests the following definition.

Definition 1. A group G is called $\diamond$-semiprime (or semiprime), if the following equivalent conditions hold:

(1) $x \diamond x \neq 1$ for any non-trivial $x \in G$;
(2) there are no nilpotent elements in G;
(3) there are no normal nilpotent subgroups in G.

It is easy to see that every domain is semiprime, as well as a direct product of domains. But a subgroup of a semiprime group need not to be semiprime.

The following result justifies the name of $S^\perp$ by showing that $S^\perp$ is a unique maximal normal direct complement of $\mathrm{gp}(S)$ in G.

Lemma 2. *Let G be a semiprime group. Then for any $S \subset G$ the following conditions hold*:

(1) $\mathrm{gp}(S, S^\perp) = \mathrm{gp}(S) \times S^\perp$;
(2) $\mathrm{gp}_G(S, S^\perp) = \mathrm{gp}_G(S) \times S^\perp$;
(3) *if* $\mathrm{gp}(S, A) = \mathrm{gp}(S) \times A$ *for some normal subgroup $A \leqslant G$, then $A \leqslant S^\perp$.*

Proof. Let $S \subseteq G$. By Lemma 1 the complement $S^\perp$ is a normal subgroup of G. From the definition of the $\diamond$-product follows that $[\mathrm{gp}_G(S), S^\perp] = 1$. Since G is semiprime there are no non-trivial elements $x \in G$ with $x \diamond x = 1$, hence $\mathrm{gp}_G(S) \cap S^\perp = 1$. This shows (1) and (2). To see (3) it suffices to notice that if $[S, A] = 1$ for a subset $A \subseteq G$ then $A \subseteq S^\perp$. $\square$

Recall that a system $E = \{e_1, \ldots, e_m\} \subset G$ is *orthogonal* if $e_i \neq 1$ and $e_i \diamond e_j = 1$ for all $1 \leqslant i \neq j \leqslant n$.

An orthogonal system $E \subset G$ is called *maximal* if $E^\perp = 1$, it is called *reduced* if every element of E is reduced, i.e., it is not a product of two non-trivial orthogonal elements. By the Zorn's lemma every (reduced) orthogonal system of a group G is contained in a maximal (reduced) orthogonal system.

Now, following classical ring theory, we develop an approach to direct decompositions of semiprime groups via orthogonal systems.

Let

$$G = G_1 \times \cdots \times G_n \tag{6}$$

be a direct product of groups. By $\pi_i : G \to G_i$ we denote the canonical projection $(g_1, \ldots, g_n) \to g_i$. Sometimes we identify the group G_i with its image in G under the canonical embedding $g_i \to (1, \ldots, g_i, \ldots, 1)$. A direct decomposition $G = G_1 \times \cdots \times G_n$ is called *reduced* if each G_i is a non-trivial directly indecomposable group. We say that G has a unique (up to a permutation of factors) direct decomposition (6) if for any other reduced direct decomposition $G = H_1 \times \cdots \times H_m$ one has $m = n$ and there is a permutation $\sigma \in \mathrm{Sym}(n)$ such that $G_i = H_{\sigma(i)}$ for every $i = 1, \ldots, n$.

For an element $g \in G$ by $\mathrm{supp}(g)$ we denote the support of g, i.e., the set $\{i \mid \pi_i(g) \neq 1\}$.

Proposition 1. *Let $G = G_1 \times \cdots \times G_n$ be a finite direct product of domains $G_1, \ldots, G_n$. Then the following hold*:

(1) *Elements $g, h \in G$ are orthogonal if and only if* $\mathrm{supp}(g) \cap \mathrm{supp}(h) = \emptyset$.
(2) *A system $E \subset G$ is maximal reduced orthogonal if and only if it is orthogonal and $|E| = n$. Moreover, in this event $E = \{e_1, \ldots, e_n\}$ where $1 \neq e_i \in G_i$.*

(3) *For any* $g_i \in G_i$

$$g_i^{\perp} = \mathrm{gp}(G_j \mid j \neq i), \qquad \left(g_i^{\perp}\right)^{\perp} = G_i.$$

(4) G *has a unique* (*up to a permutation of factors*) *reduced direct decomposition, moreover, it can be written as*

$$G = \left(e_1^{\perp}\right)^{\perp} \times \cdots \times \left(e_n^{\perp}\right)^{\perp},$$

where $\{e_1, \dots, e_n\}$ *is an arbitrary orthogonal system of* n *elements in* G.

Proof. (1) is obvious. It follows from (1) that any system $E = \{e_1, \dots, e_n\}$ with $1 \neq e_i \in G_i$ is orthogonal ($e_i \diamond e_i \neq 1$ since G_i is a domain). Now if $g \in E^{\perp}$ then for every i $\mathrm{supp}(g) \cup \mathrm{supp}(e_i) = \emptyset$, hence $\mathrm{supp}(g) = \emptyset$, i.e., $g = 1$. This shows that E is maximal. To see that each e_i is reduced (not a product of two non-trivial orthogonal elements) it suffices to notice that if $x \perp y$ then $\mathrm{supp}(xy) = \mathrm{supp}(x) \cup \mathrm{supp}(y)$. Observe also, that the argument above shows that if E is an orthogonal system of n elements then $\{\mathrm{supp}(e) \mid e \in E\}$ is a system of n disjoint subsets of $\{1, \dots, n\}$, hence $|\mathrm{supp}(e)| = 1$ for every $e \in E$, as required.

Conversely, if E is a maximal reduced orthogonal system in G, then $\{\mathrm{supp}(e) \mid e \in E\}$ is a system of disjoint subsets of $\{1, \dots, n\}$. Since E is reduced then $|\mathrm{supp}(e)| = 1$ for any $e \in E$. Indeed, let $e \in E$ and $\mathrm{supp}(e) = I \cup J$ for some non-empty and disjoint I, J. Then $e = e(I)\dot{e}(J)$ for some non-trivial $e(I), e(J)$ with $\mathrm{supp}(e(I)) = I$, $\mathrm{supp}(e(J)) = J$—contradicting to the condition that e is reduced. This shows that $|\mathrm{supp}(e)| = 1$ and in view of maximality of E the condition (2) holds.

To see (3) fix an element $1 \neq g_i \in G_i$ and notice that $G_j \subset g_i^{\perp}$ for every $j \neq i$. Since $g_i^{\perp}$ is a subgroup of G it follows that $\mathrm{gp}(G_j \mid j \neq i) \subset g_i^{\perp}$. Now, if $\mathrm{gp}(G_j \mid j \neq i) \neq g_i^{\perp}$ then there exists a non-trivial element $f \in G_i \cap g_i^{\perp}$. It follows that $f \diamond g_i = 1$—contradiction with the condition that G_i is a domain. This proves the first equality in (3), a similar argument proves the second one.

(4) follows from (2) and (3). Indeed, let E be a maximal reduced orthogonal system in G (it exists by Zorn's lemma). It follows from (2) that any reduced direct decomposition of G has precisely $|E|$ factors. Moreover, each e from E belongs to one and only one factor from a given reduced decomposition of G and by (3) that factor is equal to $(e^{\perp})^{\perp}$.

This proves the proposition. □

Notation. Let D_k be the class of groups which are direct products of k non-trivial domains, and

$$D_\omega = \bigcup_k D_k.$$

By Proposition 1 for a group $G \in D_\omega$ the reduced direct decomposition $G = G_1 \times \cdots \times G_k$ is unique (up to an ordering of factors). We will refer to these factors G_i as to *components* of G. By $\mathrm{comp}(G)$ we denote the number of components of G.

Now we are ready to discuss model theoretic properties of direct products of domains.

Lemma 3. *For every positive integer k there exists a universal formula $\mathrm{Ort}_k(x_1, \ldots, x_k)$ such that for a group G and a k-tuple $E \in G^k$ the formula $\mathrm{Ort}_k(E)$ holds in G if and only if E is an orthogonal system in G.*

Proof. Set

$$\mathrm{Ort}_2(x_1, x_2) = \forall y \quad \left(\left[x_1{}^y, x_2\right] = 1 \wedge x_1 \neq 1 \wedge x_2 \neq 1\right).$$

For any group G if the formula $\mathrm{Ort}_2(x_1, x_2)$ holds on $g, h \in G$ then g and h are non-trivial and $g \perp h$. Now for $k \geqslant 3$ put

$$\mathrm{Ort}_k(x_1, \ldots, x_k) = \bigwedge_{1 \leqslant i < j \leqslant k} \mathrm{Ort}_2(x_i, x_j).$$

Obviously, Ort_k holds on elements $g_1, \ldots, g_k \in G$ if and only if $\{g_1, \ldots, g_k\}$ is an orthogonal system in G. This proves the lemma. □

The following result shows that for each group $G \in D_k$ the set of elements g with $|\mathrm{supp}(g)| = 1$ is definable in G, as well as each component of G.

Lemma 4. *Let k be a positive integer. Then there exists a formula $\mathrm{Comp}_k(x, p)$ and a formula $P_k(p)$ such that for each group $G \in D_k$ the following conditions hold*:

(1) *for any $g \in G$*

$$G \models P_k(g) \quad \Longleftrightarrow \quad \left|\mathrm{supp}(g)\right| = 1;$$

(2) *for any $g \in G$ with $|\mathrm{supp}(g)| = 1$ the truth set $\mathrm{Comp}(G, g)$ of the formula $\mathrm{Comp}(x, g)$ coincides with the component G_g of G containing g.*

Proof. Let

$$P_k(p) = \exists x_2 \ldots \exists x_k \, \mathrm{Ort}_k(p, \ldots, x_k).$$

Then, in view of Lemma 3, $P_k(g)$ holds on $g \in G$ if and only if g is a part of an orthogonal system of k elements. Hence, by Proposition 1 $|\mathrm{supp}(g)| = 1$, as required.

To show (2) put

$$\mathrm{Comp}_k(x, p) = \forall y \quad (y \diamond p = 1 \rightarrow x \diamond y = 1),$$

where $y \diamond p = 1$ is viewed as the formula $\forall z[y, z^{-1}pz] = 1$, and similarly for $x \diamond y$. Clearly, the truth set $\mathrm{Comp}_k(G, g)$ of the formula $\mathrm{Comp}_k(x, g)$ coincides with $(g^{\perp})^{\perp}$, which is equal, by Proposition 1, to the component of G containing g. This proves (2) and the lemma. □

Now we are ready for the proof of Theorem A from the introduction.

Theorem A. *Let G be a finite direct product of domains. Then the elementary theory of each component of G is interpretable in the group G.*

Proof. Let $G \in D_k$ and $G = G_1 \times \cdots \times G_k$ be its component decomposition. By Lemma 4 there exist formulas $P_k(p)$ and $\mathrm{Comp}_k(x, p)$ such that for any $g \in G$ for which $P_k(g)$ holds in G the formula $\mathrm{Comp}_k(x, g)$ with the parameter g defines a component of G, containing g. In particular, every component of G occurs as the truth set of $\mathrm{Comp}_k(x, g)$ for some g. By Lemma 1 for an arbitrary $g \in G$ the formula $\mathrm{Comp}(x, g)$ defines a subgroup (perhaps, trivial) of G. This shows that the formula $\mathrm{Comp}_k(x, p)$ gives rise to a group code (see Section 1.1)

$$C = \bigl\{\mathrm{Comp}_k(x, p), E(x, y, p), \mathrm{Mult}(x, y, z, p), \mathrm{Inv}(x, y, p)\bigr\}$$

in which $E(x, y, p)$ is the standard equality in G and the formulas Mult, Inv are the multiplication and the inversion in G. To show that for every component G_i its elementary theory $\mathrm{Th}(G_i)$ is interpretable in G it suffices to construct a formula $P_{ki}(p)$ such that for every $g \in G$ if $P_{ki}(g)$ holds in G then the code C with the parameter g interprets in G a component G_j with the same elementary theory as the given G_i, i.e., $\mathrm{Th}(C(G, g)) = \mathrm{Th}(G_i)$. To this end, fix a component G_i of G and consider the set of indices

$$J_i = \bigl\{j \bigm| 1 \leqslant j \leqslant k,\ \mathrm{Th}(G_j) \neq \mathrm{Th}(G_i)\bigr\}.$$

Then for every $j \in J_i$ there exists a sentence ϕ_{ij} such that $\phi_{ij} \in \mathrm{Th}(G_i)$, but $\phi_{ij} \notin \mathrm{Th}(G_j)$. Put

$$\psi_i = \bigwedge_{j \in J_i} \phi_{ij}.$$

Clearly, ψ_i holds in a component G_m if and only if $\mathrm{Th}(G_m) = \mathrm{Th}(G_i)$. By the fundamental property of interpretations (Section 1.1) for every g satisfying $P_k(g)$ the translation $T_C(\psi_i)(g)$ holds in G if and only if ψ_i holds in $C(G, g)$. This implies that the formula

$$P_{ki}(p) = P_k(p) \wedge T_C(\psi_i)(p)$$

holds on an element $g \in G$ if and only if the code C with the parameter g interprets in G a component with the same elementary theory as of G_i. Therefore, the elementary theory of each component of G is interpretable in G. This proves the theorem. □

Corollary 1. *Let $G \in D_k$ and*

$$G = G_1 \times \cdots \times G_k$$

be its component decomposition. Then the following hold:

(1) $\mathrm{Th}(G)$ *is decidable if and only if* $\mathrm{Th}(G_i)$ *is decidable for every* $i = 1, \ldots, k$;
(2) $\mathrm{Th}(G)$ *is* λ*-stable if and only if* $\mathrm{Th}(G_i)$ *is* λ*-stable for every* $i = 1, \ldots, k$.

Proof. Let $\mathrm{Th}(G)$ be decidable. By Theorem A, the elementary theory $\mathrm{Th}(G_i)$ is interpretable in G for each component G_i by the group code C and the formula P_{ki} (see the argument in the proof of the theorem). Then from the fundamental property of interpretations we see that for any sentence ϕ

$$G_i \models \phi \quad \Longrightarrow \quad G \models T_C(\phi).$$

Since the translation T_C is an effective map the elementary theory $\mathrm{Th}(G_i)$ is also decidable. Conversely, if every component G_i has a decidable elementary theory then the elementary theory of their finite direct product $G = G_1 \times \cdots \times G_k$ is also decidable. This is due to S. Feferman and R. Vaught [3]. This proves (1). The proof of the statement (2) is similar and we omit it. □

Our next result shows that the number of components of a group from D_ω is also a logical invariant of the group.

Proposition 2. *For every positive integer k the class D_k is finitely axiomatizable.*

Proof. We use notations from Theorem A. For $k = 1$ put

$$\mathcal{A}_1 = \forall x \forall y \exists z \quad \bigl(x \neq 1 \wedge y \neq 1 \to \bigl[x, y^z\bigr] \neq 1\bigr).$$

Clearly, $\mathcal{A}_1$ axiomatizes the class of all domains D_1.

Let $k \geqslant 2$. Denote by $\mathcal{A}_k$ a first-order sentence in group theory language which says that there are elements $e_1, \ldots, e_k \in G$ such that the following conditions hold:

(a) The system $E = \{e_1, \ldots, e_k\}$ is an orthogonal system in G (one needs the formula $\mathrm{Ort}_k(x_1, \ldots, x_k)$ from Lemma 3 to write down this condition).
(b) For every $e_i \in E$ the set $(e_i^\perp)^\perp$ is a normal subgroup of G (can be easily done using the formula $\mathrm{Comp}_k(x, p)$ from Lemma 4). Denote this subgroup by G_i.
(c) $G = G_1 \times \cdots \times G_k$. To write down this condition by a formula it suffices to notice that since the subgroups G_i are normal in G the following equalities hold for each $i = 1, \ldots, k$:

$$\mathrm{gp}(G_j \mid j \neq i) = G_1 \ldots G_{i-1} G_{i+1} \ldots G_k.$$

Indeed, now one can easily write down that

$$G_i \cap \mathrm{gp}(G_j \mid j \neq i) = 1 \quad \text{and} \quad G = G_1 \ldots G_k.$$

(d) G_i is a domain for every $i = 1, \ldots, k$. This is equivalent to the condition that $\mathcal{A}_1$ holds in each G_i. Observe, that the translation $T_C(\mathcal{A}_1)(g)$ holds in G if and only if $\mathcal{A}_1$ holds in the interpretation $C(G, g)$. Hence, it suffices to write down the conjunction of the formulas $T_C(\mathcal{A}_1)(e_i)$ for every $e_i \in E$.

Clearly, a group G belongs to D_k if and only if G satisfies the axiom $\mathcal{A}_k$. □

Now we can describe arbitrary groups which are elementary equivalent to a given group from D_k.

Corollary 2. *Let G, H be groups and $G \in D_k$. Then $G \equiv H$ if and only if $H \in D_k$ and $G_i \equiv H_i$, where G_i, H_i are components of G and H in a suitable enumeration.*

Proof. The result follows from Theorem A, Proposition 2, and the fundamental property of interpretations. $\square$

Notice that Corollary A from the introduction summarizes the results from Corollaries 1 and 2.

Remark 1. One can generalize some of the results above to the case when $G = H \times C$, where $H \in D_k$ and C is an abelian group.

Indeed, in this case C is the center of G, hence it is definable in G, as well as the quotient group $G/C \simeq H$. We leave details to the reader.

3. Subdirect products

In this section we generalize results from Section 2 to subdirect products of domains. Throughout this section we continue to use notations from the previous sections.

Let $G = G_1 \times \cdots \times G_k$ be a direct product of groups G_i. Recall, that a subgroup H of G is called a *subdirect product* of groups G_i if $\pi_i(H) = G_i$ for every $i = 1, \dots, n$.

An embedding

$$\phi : H \hookrightarrow G_1 \times \cdots \times G_k \tag{7}$$

is called a *subdirect decomposition* of H if $\phi(H)$ is a subdirect product of the groups G_i. Sometimes, we identify H with $\phi(H)$ along ϕ, and G_i with its canonical image in $G = G_1 \times \cdots \times G_k$. The subdirect decomposition (7) termed *minimal* if $H \cap G_i \neq \{1\}$ for every $i = 1, \dots, n$ (here G_i and H are viewed as subgroups of G).

The following simple lemma shows that given a subdirect decomposition of H one can obtain a minimal one by deleting non-essential factors.

Lemma 5. *Let $\phi : H \hookrightarrow G_1 \times \cdots \times G_k$ be a subdirect decomposition of a group H. Then there is a subset $J \subset \{1, \dots, k\}$ and an embedding $\phi^* : H \hookrightarrow \prod_{j \in J} G_j$ such that ϕ^* is a minimal subdirect decomposition of H.*

Proof. Let I be a maximal subset of $\{1, \dots, k\}$ such that

$$H \cap \prod_{i \in I} G_i = \{1\}.$$

Then the following composition of homomorphisms

$$H \overset{\phi}{\hookrightarrow} \prod_{i=1}^{k} G_i \to \prod_{i=1}^{k} G_i \Big/ \prod_{i\in I} G_i \simeq \prod_{j\notin I} G_j$$

gives rise to the required embedding ϕ^*.

Let H be a subgroup of G. For elements $x, y \in H$ we have two different types of $\diamond$-products, with respect to the groups H and G:

$$x \diamond_H y = \big[\mathrm{gp}_H(x), \mathrm{gp}_H(y)\big], \qquad x \diamond_G y = \big[\mathrm{gp}_G(x), \mathrm{gp}_G(y)\big].$$

We use subscripts to notify in which group the corresponding object takes place and use this approach in all other similar circumstances (for example, $x \perp_H y$, or $x \perp_G y$). $\square$

Lemma 6. *Let $H \leqslant G_1 \times \cdots \times G_k$ be a subdirect product of groups $G_1, \ldots, G_k$. Then for elements $x, y \in H$ the following equivalence holds*:

$$x \diamond_H y = 1 \iff x \diamond_G y = 1.$$

Proof. Put $I = \{1, \ldots, k\}$. Since H is a subdirect product of groups $G_1, \ldots, G_k$ for any $h \in H$ and $i \in I$ one has

$$\pi_i\big(\mathrm{gp}_H(h)\big) = \mathrm{gp}_{G_i}\big(\pi_i(h)\big).$$

It follows that for any $x, y \in H$,

$$\begin{aligned}
x \diamond_H y = 1 \quad &\iff \quad \big[\mathrm{gp}_H(x), \mathrm{gp}_H(y)\big] = 1 \\
&\iff \quad \forall i \in I \quad \pi_i\big(\big[\mathrm{gp}_H(x), \mathrm{gp}_H(y)\big]\big) = 1 \\
&\iff \quad \forall i \in I \quad \big[\pi_i\big(\mathrm{gp}_H(x)\big), \pi_i\big(\mathrm{gp}_H(y)\big)\big] = 1 \\
&\iff \quad \forall i \in I \quad \big[\mathrm{gp}_{G_i}\big(\pi_i(x)\big), \mathrm{gp}_{G_i}\big(\pi_i(y)\big)\big] = 1 \\
&\iff \quad \forall i \in I \quad \pi_i(x) \diamond_{G_i} \pi_i(y) = 1 \\
&\iff \quad x \diamond_G y = 1.
\end{aligned}$$

This proves the lemma. $\square$

Proposition 3. *Let $G = G_1 \times \cdots \times G_k$ be a direct product of non-trivial domains and $H \hookrightarrow G_1 \times \cdots \times G_k$ be a minimal subdirect decomposition of a group H. Then the following hold*:

(1) *for $x, y \in H$, $x \perp_H y \Leftrightarrow \mathrm{supp}(x) \cap \mathrm{supp}(y) = \emptyset$*;
(2) *let $E \subset H$ be an orthogonal system in H. Then $|E| \leqslant k$ and $|E| = k$ if and only if $E = \{e_1, \ldots, e_k\}$ where $1 \neq e_i \in H \cap G_i$*;

(3) *for any* $h_i \in H \cap G_i$,

$$h_i^{\perp_H} = H \cap \ker \pi_i, \qquad H/h_i^{\perp_H} \simeq G_i, \qquad \left(h_i^{\perp_H}\right)^{\perp_H} = H \cap G_i;$$

(4) H *has a unique* (*up to a permutation of factors*) *minimal subdirect decomposition into a product of domains. Moreover, it can be written as*

$$H \hookrightarrow H/h_1^{\perp_H} \times \cdots \times H/h_k^{\perp_H},$$

where $\{h_1, \ldots, h_k\}$ *is an arbitrary orthogonal system of* k *elements in* H.

Proof. It follows from Proposition 1 and Lemma 6. □

Notation. Denote by SD_k the class of groups which are minimal subdirect products of k domains, and put $SD_\omega = \bigcup_k SD_k$.

By Proposition 3 a group $H \in SD_\omega$ has a unique (up to a permutation of factors) minimal subdirect decomposition $H \hookrightarrow G_1 \times \cdots \times G_k$ into a product of domains. We will refer to these factors G_i as to *components* of H.

Lemma 7. *A group* $G \in SD_\omega$ *has exactly* k *components if and only if* G *satisfies the sentence* $\exists X_k \operatorname{Ort}_k(X_k) \wedge \neg(\exists X_{k+1} \operatorname{Ort}_{k+1}(X_{k+1}))$.

Proof. Follows from Lemma 3 and Proposition 3. □

Theorem B. *Let* H *be a minimal subdirect product of domains. Then the elementary theory of each component of* H *is interpretable in the group* H.

Proof. Note that for any $h_i \in H \cap G_i$ the normal subgroup

$$h_i^{\perp_H}\left\{x \in H \mid \forall v\left(\left[x, h_i^v\right] = 1\right)\right\}$$

is definable in H. Hence the factor-group $H/h_i^{\perp_H}$ is interpretable in H (see Section 1.1). The rest of the proof is similar to that one in Theorem A.

From the properties of interpretations we deduce similar to the case of direct decompositions (see Corollary 1) the following results. □

Corollary 3. *Let* $H \in SD_k$ *and*

$$H \hookrightarrow G_1 \times \cdots \times G_k$$

be its minimal component decomposition. Then the following hold:

(1) *if* $\operatorname{Th}(H)$ *is decidable then* $\operatorname{Th}(G_i)$ *is decidable for every* $i = 1, \ldots, k$;
(2) *if* $\operatorname{Th}(H)$ *is* λ*-stable then* $\operatorname{Th}(G_i)$ *is* λ*-stable for every* $i = 1, \ldots, k$.

Theorem 4.

(1) *For every positive integer k class SD_k is finitely axiomatizable.*
(2) *Let H, K be groups and $H \in SD_k$. If $K \equiv H$ then $K \in SD_k$ and $H_i \equiv K_i$, where H_i, K_i are components of H and K in a suitable enumeration.*

Proof. (1) There exists a first order sentence B_k in the group theory language which holds in a group H if and only if there are elements $h_1, \ldots, h_k \in H$ such that the following conditions hold:

(a) The system $E = \{h_1, \ldots, h_k\}$ is a maximal orthogonal system in H (one can use the formula Ort_k to write down this condition).
(b) For every $h_i \in E$ the set $h_i^\perp$ is a normal subgroup of H (obvious formula).
(c) $H/h_i{}^\perp$ is a domain for every $h_i \in E$ (by Theorem B the group $H/h_i{}^\perp$ is interpretable in H. Since domains are axiomatic one can use the fundamental property of the interpretations to write down this condition).
(d) $h_1^\perp \cap \cdots \cap h_k^\perp = 1$ (obvious formula). Clearly if $H \models B_k$, then

$$H \hookrightarrow H/h_1^\perp \times \cdots \times H/h_l^\perp,$$

hence $H \in SD_k$.

(2) The result follows from statement (1), Theorem B, the fundamental property of interpretations, and the fact that the corresponding components of H and K are interpretable in H and K by the same codes. □

4. Irreducible components of algebraic sets

In this section we apply the technique of orthogonal systems to coordinate groups of algebraic sets over equationally Noetherian non-abelian CSA groups.

Theorem 5. *Let G be an equationally Noetherian non-abelian CSA-group, and Y be an algebraic set over G. Then the following conditions hold*:

(1) *the number of irreducible components of Y is equal to k if and only if $\Gamma(Y)$ satisfies the formula $\exists X\, \mathrm{Ort}_k(X)$ and does not satisfy the formula $\exists X\, \mathrm{Ort}_{k+1}(X)$;*
(2) *the coordinate group $\Gamma(Y_i)$ of each irreducible component Y_i of Y is interpretable in the group $\Gamma(Y)$;*
(3) *the elementary theory $\mathrm{Th}(\Gamma(Y_i))$ of each irreducible component Y_i of Y is interpretable in the group $\Gamma(Y)$.*

Proof. Let $Y = Y_1 \cup \cdots \cup Y_k$ be a decomposition of Y as a union of irreducible components. By Theorem 2 (see Section 1.4) the coordinate group $\Gamma(Y)$ is a minimal subdirect product of the coordinate groups $\Gamma(Y_1), \ldots, \Gamma(Y_k)$. Every group $\Gamma(Y_i)$ is universally equivalent to G [1], therefore it is a non-abelian *CSA*-group, hence a domain. Now (1)–(3) follow from Theorem B, Lemma 7, and Proposition 3. This proves the theorem. □

Observe, that Theorem C from the introduction is just a part of Theorem 5.

Corollary 4. *Let G be an equationally Noetherian non-abelian CSA group and Y be an algebraic set over G. Then the following conditions hold*:

(1) *if* $\mathrm{Th}(\Gamma(Y))$ *is decidable, then* $\mathrm{Th}(\Gamma(Y_i))$ *is decidable for every irreducible component Y_i of Y*;
(2) *if* $\mathrm{Th}(\Gamma(Y))$ *is λ-stable, then* $\mathrm{Th}(\Gamma(Y_i))$ *is λ-stable for every irreducible component Y_i of Y.*

5. Universal subclasses of qvar(F)

Recall that a group is *commutative transitive* if it satisfies the following axiom:

$$CT = \forall x, y, z \quad \big(x \neq 1 \wedge [y, x] = 1 \wedge [z, x] = 1 \to [y, z] = 1\big).$$

Let $X_n = \{x_{11}, x_{12}, \ldots, x_{n1}, x_{n2}\}$. Consider the following open formulas:

$$\Phi_n(X_n) = \bigwedge_{i=1}^{n} \big([x_{i1}, x_{i2}] \neq 1\big) \bigwedge_{i\neq j=1}^{n} \left(\bigwedge_{k,l=1}^{2} [x_{ik}, x_{jl}] = 1 \right),$$

$$\Psi_n(X_n, z) = \Phi_n(X_n) \bigwedge_{i=1}^{n} \left(\bigwedge_{l=1}^{2} [z, x_{il}] = 1 \right).$$

Lemma 8. *Let $G = G_1 \times \cdots \times G_k$ be a direct product of non-trivial commutative–transitive groups. Then the following holds*:

(1) *G satisfies the existential formula $\exists X_n \Phi_n(X_n)$ if and only if at least n of the groups $G_1, \ldots, G_k$ are non-abelian*;
(2) *G satisfies the existential formula $\exists X_n \exists z \Psi_n(X_n, z)$ if and only if at least n of the groups $G_1, \ldots, G_k$ are non-abelian, and at least one of them is abelian.*

Proof. We start with the following.

Claim 1. *Let $G \models \Phi_2(u_1, u_2, v_1, v_2)$ for some elements $u_1, u_2, v_1, v_2 \in G$. Then* $\mathrm{supp}([u_1, u_2]) \cap (\mathrm{supp}(v_1) \cup \mathrm{supp}(v_2)) = \emptyset$.

Indeed, since $[u_1, u_2] \neq 1$ then $\operatorname{supp}([u_1, u_2]) \neq \emptyset$. Let $i \in \operatorname{supp}([u_1, u_2])$. If $i \in \operatorname{supp}(v_1)$ then the following holds in the group G_i:

$$\big[\pi_i(u_1), \pi_i(u_2)\big] \neq 1, \qquad \big[\pi_i(u_1), \pi_i(v_1)\big] = 1, \qquad \big[\pi_i(u_2), \pi_i(v_1)\big] = 1,$$
$$\pi_i(v_1) \neq 1.$$

This contradicts to the condition that G_i is commutative–transitive. Hence $i \notin \operatorname{supp}(v_1)$. Similarly, $i \notin \operatorname{supp}(v_2)$. The claim follows.

Notice now, that if, say, the groups $G_1, \ldots, G_n$ are non-abelian then the set of elements $U_n = \{u_{11}, u_{12}, \ldots, u_{n1}, u_{n2}\}$ such that $u_{i1}, u_{i2} \in G_i$ and $[u_{i1}, u_{i2}] \neq 1$, satisfies $\Phi_n(X_n)$ in G.

Conversely, suppose a set of elements U_n from G satisfies $\Phi_n(X_n)$ in G. Take any $i_m \in \operatorname{supp}([u_{m1}, u_{m2}])$. By the claim above $i_m \notin \operatorname{supp}(u_{jl})$ for every $j \neq m$ and $l = 1, 2$. In particular, $i_m \notin \operatorname{supp}([u_{j1}, u_{j2}])$. It implies that the groups $G_{i_1}, \ldots, G_{i_n}$ are non-abelian, as required. This proves (1). The statement (2) easily follows from (1). $\square$

Let F be a non-abelian free group. For a non-negative integer l put

$$G_{l,0} \cong \underbrace{F \times \cdots \times F}_{l}, \qquad G_{l,1} \cong \underbrace{F \times \cdots \times F}_{l} \times \mathbf{Z}.$$

Obviously, Lemma 8 implies the following result.

Corollary 5.

$$G_{n,i} \equiv_\forall G_{m,j} \iff m = n \textit{ and } i = j.$$

Theorem 6. *Let H be a finitely generated group from* $\operatorname{qvar}(F)$. *Then the following holds*:

(1) *if $Z(H) = 1$, then $H \equiv_\forall G_{l,0}$ for some positive integer l*;
(2) *if $Z(H) \neq 1$, then $H \equiv_\forall G_{l,1}$ for some positive integer l.*

Proof. By Theorem 3, the group H is a coordinate group $\Gamma(Y)$ of an algebraic set Y defined by a coefficient-free system of equations over F. Since F is equationally Noetherian the set Y is a finite union of its irreducible components $Y = Y_1 \cup \cdots \cup Y_l$. As we have seen above, in this case $\Gamma(Y)$ is a minimal subdirect product of $\Gamma(Y_1) \times \cdots \times \Gamma(Y_l)$. This implies that $H_i = H \cap \Gamma(Y_i)$ is a non-trivial subgroup of H and $H \geqslant H_1 \times \cdots \times H_l$.

Now suppose that $Z(H) = 1$. In this event each group $\Gamma(Y_i)$ is non-abelian (otherwise, $H_i \leqslant Z(H)$), hence it contains a subgroup which isomorphic to F. Now by Theorem 3 the coordinate group $\Gamma(Y_i)$ is universally equivalent to the free group F, so it is a non-abelian CSA group. Observe, that H_i is a normal subgroup of a non-abelian CSA group $\Gamma(Y_i)$. It implies that H_i is also non-abelian. Hence, H_i contains a copy of F as a subgroup. This shows that H contains the direct product $G_{l,0}$ of l copies of F. Furthermore,

$$G_{l,0} \leqslant H \leqslant \Gamma(Y_1) \times \cdots \times \Gamma(Y_l) \equiv_\forall G_{l,0}.$$

Therefore $H \equiv_\forall G_{l,0}$. This proves (1).

Let now $Z(H) \neq 1$. If $c \in Z(H)$ and $i \in \operatorname{supp}(c)$ then $1 \neq \pi_i(c) \in Z(\Gamma(Y_i))$, hence $\Gamma(Y_i)$ is abelian. Therefore, the group H_i is abelian if and only if $\Gamma(Y_i)$ is abelian. Let $\Gamma(Y_1), \ldots, \Gamma(Y_k)$ be the only non-abelian groups among all $\Gamma(Y_i)$. Put $A = \Gamma(Y_{k+1}) \times \cdots \times \Gamma(Y_l)$, so A is a torsion-free abelian group. An argument similar to the case (1) shows that

$$G_{k,0} \times \mathbf{Z} \leqslant H \leqslant \Gamma(Y_1) \times \cdots \times \Gamma(Y_k) \times A \equiv_\forall G_{k,0} \times A.$$

Thus, $H \equiv_\forall G_{k,0} \times A$. Observe, that $A \equiv_\forall \mathbf{Z}$, so

$$H \equiv_\forall G_{k,0} \times \mathbf{Z} \equiv_\forall (G_{k,1})$$

as required. $\square$

The following result implies Theorem D from the introduction.

Corollary 6. *Let H be a finitely generated group from* $\operatorname{qvar}(F)$. *Then there exists a unique group $G_{l,i}$ such that* $\operatorname{ucl}(H) = \operatorname{ucl}(G_{l,i})$.

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