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Algebraic Geometry over Groups

I. Algebraic Sets and Ideal Theory

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The object of this paper, which is the first in a series of three, is to lay the foundations of the theory of ideals and algebraic sets over groups.

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1. INTRODUCTION

1.1. *Some General Comments*

The beginning of classical algebraic geometry is concerned with the geometry of curves and their higher dimensional analogues. Today the geometry is closely linked to the ideal theory of finitely generated polynomial algebras over fields.

For some years now we have attempted to lay the foundations of an analogous subject, which we term algebraic geometry over groups, which bears a surprising similarity to elementary algebraic geometry—hence its name. In the present paper we introduce group-theoretic counterparts to algebraic sets, coordinate algebras, the Zariski topology, and various other notions such as zero-divisors, prime ideals, the Lasker−Noether decomposition of ideals as intersections of prime ideals, the Noetherian condition, irreducibility, and the Nullstellensatz. A number of interesting concepts arise, focussing attention on some fascinating new aspects of infinite groups.

The impetus for much of this work comes mainly from the study of equations over groups. We have more to say about this later in this introduction.

1.2. *The Category of G-Groups*

Our work centers around the notion of a G-group, where here G is a fixed groups. These G-groups can be likened to algebras over a unitary commutative ring, more specially a field, with G playing the role of the coefficient ring. A group H is termed a G-group if it contains a designated copy of G, which we for the most part identify with G. Notice that we allow for the possibility that $G = 1$ and also that $G = H$; in particular G is itself a G-group. Such G-groups form a category in the obvious way. A morphism from a G-group H to a G-group H' is a group homomorphism

$\phi\colon H \to H'$ which is the identity on G. We call these morphisms *G-homomorphisms*. The kernels of G-homomorphisms are termed *ideals*; they are simply the normal subgroups which meet G in the identity. The usual notions of group theory carry over to this category, allowing us to talk about free G-groups, finitely generated, and finitely presented G-groups and so on. In particular, it is not hard to identify the finitely generated free G-groups. They take the form

$$G[X] = G[x_1, \dots, x_n] = G * F(X),$$

the free product of G and the free group $F(X) = \langle x_1, \dots, x_n \rangle$ freely generated by $\{x_1, \dots, x_n\}$. We sometimes say that $X = \{x_1, \dots, x_n\}$ freely generates the free G-group $G[X]$. We view $G[X]$ as a non-commutative analogue of a polynomial algebra over a unitary commutative ring in finitely many variables. We think of the elements of $G[X]$ as non-commutative polynomials with coefficients in G. Similarly, if we stay inside the category of G-groups, the free product of two G-groups A and B is their amalgamated product $A *_G B$, with G the amalgamated subgroup.

In dealing with various products, it is sometimes useful to let the *coefficient group* G vary. In particular, if H_i is a G_i-group for each i in some index set I, then the unrestricted direct product $\overline{\prod}_{i \in I} H_i$ can be viewed as a $\overline{\prod}_{i \in I} G_i$-group, in the obvious way. If H_i is a G-group for each i, then we sometimes think of the unrestricted direct product P of the groups H_i as a G-group by taking the designated copy of G in P to be the diagonal subgroup of the unrestricted direct product of all the copies of G in the various factors. In the case of, say, the standard wreath product, if U is a G-group and if T is a G'-group, then their (standard) wreath product $U \wr T$ can be viewed as a $G \wr G'$-group in the obvious way again.

1.3. *Notions from Commutative Algebra*

Our objective here is to introduce group-theoretic counterparts to the classical notions of integral domain and Noetherian ring.

Let H be a G-group. Then we term a non-trivial element $x \in H$ *a G-zero divisor* if there exists a non-trivial element $y \in H$ such that

$$[x, g^{-1}yg] = 1 \quad \text{for all } g \in G.$$

Notice that if $G = 1$ then every non-trivial element of the G-group H is a G-zero divisor. We then term a G-group H a *G-domain* if it does not contain any G-zero divisors; in the event that $G = H$ we simply say that H is a domain.

We focus on G-domains in Section 2.1. We recall here that a subgroup M of a group H is *malnormal* if whenever $h \in H$, $h \notin M$, then $h^{-1}Mh \cap$

$M = 1$. A group H is termed a *CSA-group* if every maximal abelian subgroup M of H is malnormal. If H is such a CSA-group and G is a non-abelian subgroup of H, then H, viewed as a G-group, is a G-domain. Notice that every torsion-free hyperbolic group is a CSA-group. This demonstrates, together with the Theorems A1–A3 below, that there is a plentiful supply of G-domains.

THEOREM A1. *If U is a domain and if T is a torsion-free domain, then the wreath product $U \wr T$ is a domain.*

Further domains can be constructed using amalgamated products.

THEOREM A2. *Let A and B be domains. Suppose that C is a subgroup of both A and B satisfying the following condition*:

$$\text{if } c \in C,\ c \neq 1, \text{ then either } [c, A] \not\subseteq C \text{ or } [c, B] \not\subseteq C. \qquad (*)$$

Then the amalgamated free product $H = A *_C B$ is a domain.

THEOREM A3. *The free product, in the category of G-groups, of two G-domains is a G-domain whenever G is a malnormal subgroup of each of the factors.*

Theorems A1–A3 are proved in Section 2.1.

The analogue of the Noetherian condition in commutative algebra in the present context is what we term *G-equationally Noetherian*. In order to explain what this means we need to digress a little. To this end, let H be a G-group. Then we term the set

$$H^n = \{(a_1, \ldots, a_n) | a_i \in H\},$$

affine n-space over H and we sometimes refer to its element as *points*. Let $G[X]$ be as in Section 1.2. As suggested earlier, the elements $f \in G[X]$ can be viewed as *polynomials* in the non-commuting *variables* $x_1, \ldots, x_n$, with *coefficients* in G. We use functional notation here,

$$f = f(x_1, \ldots, x_n) = f(x_1, \ldots, x_n, g_1, \ldots, g_m), \qquad (1)$$

thereby expressing the fact that word representing f in $G[X]$ involves the *variables* $x_1, \ldots, x_n$ and, as needed, the *constants* $g_1, \ldots, g_m \in G$. We term

$$v = (a_1, \ldots, a_n) \in H^n, \qquad (2)$$

a *root* of f if

$$f(v) = f(a_1, \ldots, a_n, g_1, \ldots, g_m) = 1.$$

We sometimes say that f *vanishes at* v. If S is a subset of $G[X]$ then v is said to be a *root of* S if it is a root of every $f \in S$; i.e., S *vanishes at* v. In this event we also say that v is an *H-point* of S. We then denote the set of all roots of S by $V_H(S)$. Then a G-group H is called G-equationally Noetherian if for every $n > 0$ and every subset S of $G[x_1, \dots, x_n]$ there exists a finite subset S_0 of S such that

$$V_H(S) = V_H(S_0).$$

In the event that $G = H$ we simply say that G is equationally Noetherian, instead of G is G-equationally Noetherian.

These G-equationally Noetherian groups are of considerable interest to us and play an important part in the theory that we develop here. We discuss them in detail in Section 2.2.

The class of all G-equationally Noetherian groups is fairly extensive. This follows from the two theorems below.

THEOREM B1. *Let a group H be linear over a commutative, Noetherian, unitary ring, e.g., a field. Then H is equationally Noetherian.*

A special case of this theorem was first proved by Bryant [BR] in 1977 and another special case, that of free groups, by Guba [GV] in 1986. Notice that the following groups are linear, hence equationally Noetherian: polycyclic [AL], finitely generated metabelian [RV2], free nilpotent, or free metabelian (Magnus, see, for example, [WB]). Not all equationally Noetherian groups are linear (see the discussion in Section 2.2).

THEOREM B2. *Let $\mathcal{E}_G$ be the class of all G-equationally Noetherian groups. Then the following hold*:

1. *$\mathcal{E}_G$ is closed under G-subgroups, finite direct products, and ultrapowers*;

2. *$\mathcal{E}_G$ is closed under G-universal (G-existential) equivalence*; *i.e., if $H \in \mathcal{E}_G$ and H' is G-universally equivalent (G-existentially equivalent) to H, then $H' \in \mathcal{E}_G$*;

3. *$\mathcal{E}_G$ is closed under G-separation*; *i.e., if $H \in \mathcal{E}_G$ and H' is G-separated by H, then $H' \in \mathcal{E}_G$.*

(We defer the definition of G-separation until Section 1.4.) Here two groups are said to be G-universally equivalent if they satisfy the same

G-universal sentences. These are formulas of the type

$$\forall x_1 \cdots \forall x_n \left(\bigvee_{j=1}^{s} \bigwedge_{i=1}^{t} \left(u_{ji}\left(\bar{x}, \bar{g}_{ij} \right) = 1 \quad \text{and} \quad w_{ij}\left(\bar{x}, \bar{f}_{ij} \right) \neq 1 \right) \right),$$

where $\bar{x} = (x_1, \ldots, x_n)$ is an n-tuple of variables, $\bar{g}_{ij}$ and $\bar{f}_{ij}$ are arbitrary tuples of elements (constants) from G. Definition of G-separation is given in the next section. We prove Theorems B1 and B2 in Section 2.2.

It is not hard to construct examples of G-groups that are not G-equationally Noetherian (see [BMRO] and Section 2.2). We note also here two theorems proved in [BMRO], namely, that if a G-group H contains an equationally Noetherian subgroup of finite index, then H itself is equationally Noetherian; and if H is an equationally Noetherian group and Q is a normal subgroup of H which is a finite union of algebraic sets in H (in particular, if Q is a finite normal subgroup or the center of H), then H/Q is equationally Noetherian. Finally, taking here for granted the terminology described in [BMR1], we remark that if G is an equationally Noetherian torsion-free hyperbolic group and if A is an unitary associative ring of characteristic zero of Lyndon's type, then the completion G^A is G-equationally Noetherian.

1.4. *Separation and Discrimination*

We concern ourselves next with specific approximation techniques in groups and rings. Two notions play an important part here, namely, that of separation and discrimination.

Let H be a G-group. Then we say that a family

$$\mathscr{D} = \{D_i | i \in I\},$$

of G-groups *G-separates* the G-group H, if for each non-trivial $h \in H$ there exists a group $D_i \in \mathscr{D}$ and a G-homomorphism $\phi: H \to D_i$ such that $\phi(h) \neq 1$.

Similarly, we say that $\mathscr{D}$ *G-discriminates* H, if for each finite subset $\{h_1, \ldots, h_n\}$ of non-trivial elements of H there exists a $D_i \in \mathscr{D}$ and a G-homomorphism $\phi: H \to D_i$ such that $\phi(h_j) \neq 1$, $j = 1, \ldots, n$.

If $\mathscr{D}$ consists of the singleton D, then we say that D *G-separates* H in the first instance and that D *G-discriminates* H in the second. If G is the trivial group, then we simply say that D separates H or D discriminates H. In this event the notions of separation and discrimination are often expressed in the group-theoretical literature by saying, respectively, that H is *residually* $\mathscr{D}$ and that H is *fully residually* $\mathscr{D}$ or H is *ω-residually* $\mathscr{D}$.

In Section 2.3 we prove, among other things, the following two useful criteria.

THEOREM C1 [BMR2]. *Let G be a domain. Then a G-group H is G-discriminated by G if and only if H is a G-domain and H is G-separated by G.*

B. Baumslag introduced and exploited this idea in the case of free groups [BB].

We say that the G-group H is *locally G-discriminated* by the G-group H' if every finitely generated G-subgroup of H is G-discriminated by H'.

THEOREM C2. *Let H and H' be G-groups and suppose that at least one of them is G-equationally Noetherian. Then H is G universally equivalent to H' if and only if H is locally G-discriminated by H' and H' is locally G-discriminated by H.*

The idea to tie discrimination to universal equivalence is due to Remeslennikov [RV1], who formulated and proved a version of Theorem C2 in the case of free groups.

1.5. *Ideals*

As usual, the notion of a domain leads one to the notion of a prime ideal. An ideal P of the G-group H is said to be a *prime ideal* if H/P is a G-domain. Prime ideals are especially useful in describing the ideal structure of an arbitrary G-equationally Noetherian G-domain H.

An ideal Q of the G-group H is termed *irreducible* if $Q = Q_1 \cap Q_2$ implies that either $Q = Q_1$ or $Q = Q_2$, for any choice of the ideals Q_1 and Q_2 of H. Irreducibility is important in dealing with ideals of a free G-group $G[X]$. In the theory that we are developing here, we define, by analogy with the classical case, the *Jacobson G-radical* $J_G(H)$ of the G-group H to be the intersection of all maximal ideals of H with quotient G-isomorphic to G; if no such ideals exist, we define $J_G(H) = H$. Similarly, we define the *G-radical* $\mathrm{Rad}_G(Q)$ of an ideal Q of a G-group H to be the pre-image in H of the Jacobson G-radical of H/Q, i.e., the intersection of all the maximal ideals of H containing Q with quotient G-isomorphic to G.

More generally, if K is any G-group, the we define *Jacobson K-radical* $J_K(H)$ of the G-group H to be the intersection of all ideals of H with quotient G-embeddable into K; similarly we define the K-radical $\mathrm{Rad}_K(Q)$ of an ideal Q of H to be the pre-image of H of the Jacobson K-radical of H/Q. Finally, an ideal of a G-group is said to be a *K-radical ideal* if it coincides with its K-radical.

A finitely generated G-group H is called a K-*affine* G-group if $J_K(H) = 1$. The K-affine groups play an important role in the abstract characterization of coordinate groups defined over K.

All these notions are discussed in detail in Sections 4 and 5.

1.6. *The Affine Geometry of G-Groups*

Let, as in Section 1.2,

$$H^n = \{(a_1, \dots, a_n) | a_i \in H\}$$

be *affine n-space over the G-group H* and let S be a subset of $G[X]$. Then the set

$$V_H(S) = \{v \in H^n | f(v) = 1, \text{ for all } f \in S\}.$$

is termed the (*affine*) *algebraic set* over H defined by S.

We sometimes denote $V_H(\{s_1, s_2, \dots\})$ by $V_H(s_1, s_2, \dots)$.

The union of two algebraic sets in H^n is not necessarily an algebraic set. We define a topology on H^n by taking as a subbasis for the closed sets of this topology, the algebraic sets in H^n. We term this topology the *Zariski topology*. If H is a G-domain, then the union of two algebraic sets is again algebraic and so in this case the closed sets in the Zariski topology consist entirely of algebraic sets.

Then, fixing the G-group H, these algebraic sets can be viewed as the objects of a category, where morphisms are defined by polynomial maps; i.e., if $Y \subseteq H^n$, and $Z \subseteq H^p$ are algebraic sets then a map $\phi: Y \to Z$ is a *morphism* in this category (or a *polynomial map*) if there exist $f_1, \dots, f_p \in G[x_1, \dots, x_n]$ such that for any $(a_1, \dots, a_n) \in Y$:

$$\phi(a_1, \dots, a_n) = \big(f_1(a_1, \dots, a_n), \dots, f_p(a_1, \dots, a_n)\big).$$

It turns out that this category is isomorphic to a subcategory of the category of all G-groups. In order to explain more precisely what this subcategory consists of we need to turn our attention to the *ideals* of algebraic sets.

1.7. *Ideals of Algebraic Sets*

Let, as before, H be a G-group, n a positive integer, H^n affine n-space over H, and $G[X] = G[x_1, \dots, x_n]$.

Let $Y \subseteq H^n$. Then we define

$$I_H(Y) = \{f \in G[X] | f(v) = 1 \text{ for all } v \in Y\}.$$

Suppose now that S is a non-empty subset of $G[X]$ and that $Y = V(S)$. Every point $y = (y_1, \dots, y_n) \in H^n$ defines a G-homomorphism ϕ_y of $G[X]$ into H, via evaluation; i.e., by definition, if $f \in G[X]$, then $\phi_y(f) = f(y)$. It follows that

$$I_H(Y) = \bigcap_{y \in Y} \ker \phi_y.$$

Hence $I_H(Y)$ is an ideal of $G[X]$ provided only that Y is non-empty. If $Y = \emptyset$ and $G \neq 1$, then $I(Y) = G[X]$ is not an ideal. We, notwithstanding the inaccuracy, term $I(Y)$ *the ideal of* Y under all circumstances.

In the event that Y is an algebraic set in H^n, then we define the *coordinate group* $\Gamma(Y)$ of Y to be the G-group of all polynomial functions on Y. These are the functions from Y into H which take the form

$$y \mapsto f(y) \qquad (y \in Y),$$

where f is a fixed element of $G[x_1, \dots, x_n]$. It is easy to see that

$$\Gamma(Y) \simeq G[X]/I(Y).$$

The ideals $I_H(Y)$ completely characterize the algebraic sets Y over H; i.e., for any algebraic sets Y and Y' over H we have:

$$Y = Y' \quad \Leftrightarrow \quad I_H(Y) = I_H(Y').$$

Similarly, the algebraic sets Y are characterized by their coordinate groups $\Gamma(Y)$,

$$Y \simeq Y' \quad \Leftrightarrow \quad \Gamma(Y) \simeq \Gamma(Y');$$

here $\simeq$ represents isomorphism in the appropriate category (see Sections 3.1 and 3.5 for details).

Amplifying the remark above, it turns out that if H is a G-group, then the category of all algebraic sets over H is equivalent to the category of all coordinate groups defined over H, which is exactly the category of all finitely generated G-groups that are G-separated by H. The latter result comes from the abstract description of coordinate groups (see Section 5.1). We need another notion from commutative algebra in order to explain how this comes about.

The ideal Q of $G[X]$ is called H-closed if $Q = I_H(Y)$ for a suitable choice of the subset Y of H^n. We prove in Section 4.2 that the H-closed ideals of $G[X]$ are precisely the H-radical ideals of $G[X]$. Therefore, a finitely generated G-group Γ is a coordinate group of an algebraic set $Y \subseteq H^n$ (for a suitable n) if and only if $J_H(\Gamma) = 1$, which is equivalent to G-separation of Γ in H.

An elaboration of this approach yield some analogues of the Lasker–Noether theorem, which we describe below.

1.8. *The Zariski Topology of Equationally Noetherian Groups*

In the event that the G-group H is G-equationally Noetherian, it turns out (irrespective of the choice of n) that the Zariski topology satisfies the descending chain condition on closed subsets of H^n; i.e., every properly descending chain of closed subsets of H^n is finite. Indeed, we have the following important theorem, which is proved in Section 3.6.

THEOREM D1. *Let H be a G-group. Then for each integer $n > 0$, the Zariski topology on H^n is Noetherian, i.e., satisfies the descending chain condition on closed sets, if and only if H is G-equationally Noetherian.*

This implies, in particular, that every closed subset of H^n is a finite union of algebraic sets. As is the custom in topology, a closed set Y is termed *irreducible* if $Y = Y_1 \cup Y_2$, where Y_1 and Y_2 are closed sets, implies that either $Y = Y_1$ or $Y = Y_2$. So, by the remark above, every closed subset Y of H^n can be expressed as a finite union of irreducible algebraic sets:

$$Y = Y_1 \cup \cdots \cup Y_n.$$

These sets are usually referred to as the *irreducible components* of Y, which turn out to be unique. This shifts the study of algebraic sets to their irreducible components. It turns out that the irreducible ideals are the algebraic counterpart to the irreducible algebraic sets. Indeed, let H be a G-domain; then (see Section 4.3) a closed subset $Y \subseteq H^n$ is irreducible in the Zariski topology on H^n if and only if the ideal $I_H(Y)$ is an irreducible ideal in $G[X]$.

An elaboration of this approach yields a very important characterization of irreducible algebraic sets in terms of their coordinate groups.

THEOREM D2. *Let H be a G-equationally Noetherian G-domain and let Y be an algebraic set in H^n. Then the following conditions are equivalent*:

1. *Y is irreducible*;
2. *$I_H(Y)$ is a prime ideal in $G[X]$*;
3. *$\Gamma(Y)$ is G-equationally Noetherian G-domain*;
4. *$\Gamma(Y)$ is G-discriminated by H.*

In the event that $G = H$ we can add one more equivalent condition to Theorem D2, which establishes a surprising relationship between coordinate groups of irreducible algebraic sets over G and finitely generated models of the universal theory of the group G.

THEOREM D3. *Let G be an equationally Noetherian domain and let Y be an algebraic set in G^n. Then the following conditions are equivalent*:

1. *Y is irreducible*;
2. *$\Gamma(Y)$ is G-universally equivalent to G.*

Moreover, any finitely generated G-group which is G-universally equivalent to G is the coordinate group of some irreducible algebraic set over G.

1.9. *Decomposition Theorems*

The categorical equivalence, described above, between algebraic sets over the G-group H and the finitely generated G-groups which are G-separated by H, leads to an analogue of a theorem often attributed to Lasker and Noether.

THEOREM E1. *Let H be a G-equationally Noetherian G-domain. Then each H-closed ideal in $G[X]$ is the intersection of finitely many prime H-closed ideals, none of which is contained in any one of the others, and this representation is unique up to order. Conversely distinct irredundant intersections of prime H-closed ideals define distinct H-closed ideals.*

Theorem E1 has a counterpart for ideals of arbitrarily finitely generated G-groups.

THEOREM E2. *Let H be a finitely generated G-group and let K be a G-equationally Noetherian G-domain. Then each K-radical ideal in H is a finite irredundant intersection of prime K-radical ideals. Moreover, this representation is unique up to order. Furthermore, distinct irredundant intersections of prime K-radical ideals define distinct K-radical ideals.*

Theorems E1 and E2 are proved in Section 4.4. They lead to several interesting corollaries. Here we mention two of them, proved in Section 5.3, which stem from the correspondence between algebraic sets and their coordinate groups.

THEOREM F1. *Let H be a G-equationally Noetherian G-domain. If Y is any algebraic set in H^n, then the coordinate group $\Gamma(Y)$ is a subgroup of a direct product of finitely many G-groups, each of which is G-discriminated by H.*

THEOREM F2. *Let H be a non-abelian equationally Noetherian torsion-free hyperbolic group. Then every finitely generated group E which is separated by H is a subdirect product of finitely many finitely generated groups, each of which is discriminated by H.*

1.10. *The Nullstellensatz*

We need next a variation of the notion of an algebraically closed group, which is due to Scott [SW]. Here a non-trivial G-group H is termed *G-algebrically closed* if every finite set of equations and inequations of the form

$$f = 1 \quad \text{and} \quad f \neq 1 \qquad (f \in G[x_1, \dots, x_n]),$$

that can be satisfied in some G-group containing H, can also be satisfied in H. This class of G-groups play a part in the discussion that follows. Notice, in the event that $G = H$ we have the standard notion of algebraically closed group due to Scott.

Hilbert's classical Nullstellensatz is often formulated for ideals of polynomial algebras over algebraically closed fields. One such formulation asserts that every proper ideal in the polynomial ring $K[x_1, \dots, x_n]$ over an algebraically closed field K, has a root in K. It is easy to prove an analogous result for G-groups (notice that this includes the case where $G = H$, below).

THEOREM G1. *Let H be a G-algebraically closed G-group. Then every ideal in $G[X]$, which can be generated as a normal subgroup by finitely many elements, has a root in H.*

Another form of the Nullstellensatz for polynomial rings can be expressed as follows. Suppose that S is a finite set of polynomials in $K[x_1, \dots, x_n]$ and that a polynomial f vanishes at all of the zeroes of S; then some power of f lies in the ideal generated by S. With this in mind, we introduce the following definition.

Let H be a G-group and let S be a subset of $G[x_1, \dots, x_n]$. Then we say that *S satisfies the Nullstellensatz over H* if

$$I(V_H(S)) = \mathrm{gp}_{G[X]}(S),$$

where here $\mathrm{gp}_{G[X]}(S)$ denotes the normal closure in $G[X]$ of S. It follows, as in the classical case, that an ideal in $G[X]$ satisfies the Nullstellensatz over H if and only if it is H-radical. Notice that in the event that $V_H(S)$ is non-empty, then $\mathrm{gp}_{G[X]}(S)$ is actually the ideal of the G-group $G[X]$ generated by S, i.e., the smallest ideal of $G[X]$ containing S.

The following version of the Nullstellensatz then holds.

THEOREM G2. *Let H be a G-group and suppose that H is G algebraically closed. Then every finite subset S of $G[x_1, \dots, x_n]$ with $V_H(S) \neq \emptyset$, satisfies the Nullstellensatz; indeed, $I(V_H(S)) = \mathrm{gp}_{G[X]}(S)$.*

There is a simple criterion for determining whether a given set satisfies the Nullstellensatz. Indeed, suppose that H is a G-group and that $V_H(S) \neq \emptyset$, where S is a subset of $G[x_1, \dots, x_n]$. Then S satisfies the Nullstellensatz over H if and only if $G[X]/\mathrm{gp}_{G[X]}(S)$ is G-separated by H. Notice, that if the group H is torsion-free and a set $S \subseteq G[X]$ satisfies the Nullstellensatz over H, then the ideal $Q = \mathrm{gp}_{G[X]}(S)$ is isolated in $G[X]$; i.e., $f^n \in Q$ implies $f \in Q$ (for any $f \in G[X]$).

It is not easy to determine which systems of equations, e.g., over a free group, satisfy the Nullstellensatz. We discuss a few examples in Section 6.

1.11. *Connections with Representation Theory*

The set $\mathrm{Hom}(J, T)$ of all homomorphisms of a finitely generated group J into a group T has long been of interest in group theory. If J is a finite group and T is the group of all invertible $n \times n$ matrices over the field $\mathscr{C}$ of complex numbers, then the study of $\mathrm{Hom}(J, T)$ turns into the representation theory of finite groups. If T is an algebraic group over $\mathscr{C}$, then $\mathrm{Hom}(J, T)$ is an affine algebraic set, the geometric nature of which lends itself to an application of the Bass−Serre theory of groups acting on trees, with deep implications on the structure of the fundamental groups of three-dimensional manifolds.

The algebraic geometry over groups that we develop here can be viewed also as a contribution to the general representation theory of finitely generated groups. In order to explain, we work now in the category of G-groups, where G is a fixed group; notice that in the event that $G = 1$, this is simply the category of all groups.

Now let J be a finitely generated G-group, equipped with a finite generating set $\{x_1, \dots, x_n\}$, and let T be an arbitrarily chosen G-group. Express J as a G-quotient group of the finitely generated free G-group $F = G[x_1, \dots, x_n]$:

$$J \simeq F/Q.$$

Then we have seen that the set $\mathrm{Hom}_G(J, T)$ of G-homomorphisms from J into T can be parametrized by the roots of Q in T^n and hence carries with it the Zariski topology. Observe that if the Nullstellensatz applies to Q, then the coordinate group of this space can be identified with J, which explains its importance here. It is not hard to see that $\mathrm{Hom}_G(J, T)$ is independent of the choice of generating set; i.e., the algebraic sets obtained are isomorphic in the sense that we have already discussed. So $\mathrm{Hom}_G(J, T)$ is a topological invariant of the G-group J, which we refer to as the space of all representations of J in T. The group $\mathrm{Inn}_G(T)$ of G-inner automorphisms of T; i.e., those inner automorphisms of T which commute elementwise with G, induce homeomorphisms of $\mathrm{Hom}_G(J, T)$ and so we can form the quotient space of $\mathrm{Hom}(J, T)$ by $\mathrm{Inn}_G(T)$, which we term the space of inequivalent representations of J in T. This is a finer invariant than $\mathrm{Hom}(J, T)$, akin to the space of inequivalent representations or characters of representations of a finite group. In the event that J and T are algorithmically tractable and satisfy various finiteness conditions it is interesting to ask whether these spaces can be described in finite terms and, if so, whether they are computable and how they can be used to provide information about J, assuming complete knowledge of T. The obvious questions involving the various algebro-geometric properties of $\mathrm{Hom}_G(J, T)$ are again of interest here, in particular for finitely generated metabelian groups. Whether this touches on the isomorphism problem for such groups remains to be seen. We leave this line of development and the way in which our algebraic geometry over groups plays a part for another time.

1.12. *Related Work*

In 1959 Lyndon [LR1] initiated a general investigation into equations over a free group. In 1960 he described solution sets of arbitrary equations of one variable over a free group. Then in 1981 Comerford and Edmunds [CE] described solution sets of quadratic equations over a free group. In 1985 Makanin [MG] showed that there is an algorithm whereby one can decide whether an arbitrary system of equations over a free group has a solution and, furthermore, that the universal theory of a free group is decidable. Shortly after that Razborov [RA1] obtained a description of algebraic sets over a free group based on the Makanin's technique.

Bryant [BR] was the first to consider the whole collection of algebraic sets of equations in one variable over a group as a basis for Zariski topology. Subsequently this idea was taken up by Guba [GV] and Stallings [SJ].

Some of the approaches we have developed here (for example, Nullstellensatz for groups) go back to Rips, who described some of his thoughts in lectures and also in private conversations.

The modern approach to some problems in model theory (for example, to the characterization of uncountably categorical theories) rests heavily on some abstract algebraic−geometric ideas that were exploited by Hrushovski and Zilber [HZ].

Elaborating the subject from the point of view of logic, Plotkin [PB] generalized the categories of algebraic sets and the corresponding affine objects to arbitrary universal algebras.

2. NOTIONS FROM COMMUTATIVE ALGEBRA

2.1. *G-Domains*

G-domains, which were introduced in Section 1.3, play an important role in this paper. We recall first the definition of a G-zero divisor.

DEFINITION 1. Let H be a G-group. A non-trivial element $x \in H$ is called a G-zero divisor if there exists a non-trivial element $y \in H$ such that

$$[x, y^g] = 1 \quad \text{for all } g \in G. \tag{3}$$

The G-group H is termed a G-domain if it has no G-zero divisors.

In the case when $G = H$ we omit all mention of G and simply say that H is a *domain*. Similarly, if x is an element of the group H, then we say that x is a zero-divisor if it is an H-zero divisor; i.e., we view H as an H-group. Notice that if H is G-domain for some $G \leq H$, then H is also a domain.

We adopt throughout this paper the following notation. If J is a subgroup of a group K and S is a subset of K, then we denote the subgroup of K generated by the conjugates of all of the elements in S by all of the elements of J by $\mathrm{gp}_J(S)$. So $\mathrm{gp}_K(S)$ is the normal closure of S in K, i.e., the least normal subgroup of K containing S.

Notice that the equation (3) is equivalent to the equation

$$[\mathrm{gp}_G(x), \mathrm{gp}_G(y)] = 1. \tag{4}$$

We call the subgroup $[\mathrm{gp}_G(x), \mathrm{gp}_G(y)]$ the $\diamond$*-product* of x and y and denote it by $x \diamond y$. So a non-trivial element x in the G-group H is a G-zero divisor if and only if $x \diamond y = 1$ for some nontrivial $y \in H$. Obviously,

$$x \diamond x = y \diamond x; \tag{5}$$

therefore the element y in Definition 1 is also a G-zero divisor. Notice also that if x is a G-zero divisor, then all the non-trivial elements in $\mathrm{gp}_G(x)$ are also G-zero divisors.

DEFINITION 2. We say that an element $x \in H$ is G-nilpotent of degree $\leq k$ if

$$[x^{g_1}, x^{g_2}, \ldots, x^{g_k}] = 1 \quad \text{for all } g_i \in G,$$

i.e., if $\mathrm{gp}_G(x)$ is a nilpotent subgroup of class $\leq k$. In the event that $\mathrm{gp}_G(x)$ is nilpotent of class exactly k, then we say that x is G-nilpotent of degree k.

Again, if $G = H$ then we omit all mention of G and say that the element x is *nilpotent*. Notice, that if a group H is nilpotent of class c, then every element is nilpotent of degree at most c.

The following lemma shows that G-nilpotent elements are G-zero divisors.

LEMMA 1. *Every non-trivial G-nilpotent element in a G-group H is a G-zero divisor.*

Proof. Let $x \in H$ be a non-trivial G-nilpotent element. Thus $\mathrm{gp}_G(x)$ is nilpotent. If z is a non-trivial element in the center of $\mathrm{gp}_G(x)$, then $[x, z^g] = 1$ for every choice of $g \in G$, as needed.

In the case of an associative ring, invertible elements are never zero divisors. Similar result holds for groups. To this end we introduce here the following definition.

DEFINITION 3. Let H be a G-group. An element $h \in H$ is termed G-invertible if $\mathrm{gp}_G(h) \cap G \neq 1$.

The following lemma then holds.

LEMMA 2. *Let H be a G-group and assume that the subgroup G in H does not contain any G-zero divisors from H. Then any G-invertible element in H is not a G-zero divisor.*

Proof. Let x be a G-invertible element in H. So there exists a non-trivial element $g \in \mathrm{gp}_G(x) \cap G$. Suppose that x is a G-zero divisor. Then there exists $y \in H$ such that $[\mathrm{gp}_G(x), \mathrm{gp}_G(y)] = 1$. It follows that $[\mathrm{gp}_G(g), \mathrm{gp}_G(y)] = 1$; i.e., g is a G-zero divisor, a contradiction.

G-domains have a somewhat restricted normal subgroup structure, as the following lemma shows.

LEMMA 3. *Let H be a G-domain. Then the following hold*:

1. *G is a non-abelian group*;
2. *Every G-subgroup of H is a G-domain*;

3. *Every abelian normal subgroup of* H *is trivial*; *in particular*, *if* $H \neq 1$, *then* H *is not solvable and hence not nilpotent*;

4. H *is directly indecomposable*.

The proof is straightforward and is left to the reader.

Our next objective is to show that the class of G-domains is fairly extensive. As noted in the Introduction, this class contains all non-abelian G-groups which are also CSA-groups; hence it contains all non-abelian, torsion-free hyperbolic groups and all groups acting freely on Λ-trees. It is not hard to prove that in a CSA-group H the centralizers of all non-trivial elements are abelian [MR2]. This is equivalent to saying that *commutativity is a transitive* relation on the set of all non-trivial elements of H [FGMRS]. We routinely make use of this property.

PROPOSITION 1. *Let* G *be a non-abelian group and let* H *be a CSA* G*-group. Then* H *is a* G*-domain.*

Proof. Let $a, b \in G$, $[a, b] \neq 1$ and suppose x, y are non-trivial elements of H. If

$$[x, y^a] = [x, y^b] = [x, y^{ab}] = 1,$$

then by the transitivity of commutation $[y^b, y^{ab}] = 1$ and $[y^a, y^b] = 1$. The first relation implies that $[y, y^a] = 1$ and since a maximal abelian subgroup M of H containing y is malnormal in H, we have $[y, a] = 1$. Now from $[y^a, y^b] = 1$ it follows that $[y, y^b] = 1$ and consequently, $[y, b] = 1$. This implies $[a, b] = 1$, a contradiction, which completes the proof of the lemma.

It follows directly from the argument above that we have also proved the following corollary.

COROLLARY 1. *Let* H *be a* G*-group and suppose that* H *is a CSA-group. If* a *and* b *are elements of* G *and if* $[a, b] \neq 1$, *then for every choice of the non-trivial elements* x *and* y *in* H, *at least one of the following hold*

$$[x, y^a] \neq 1 \quad \text{or} \quad [x, y^b] \neq 1 \quad \text{or} \quad [x, y^{ab}] \neq 1.$$

A related result holds for G-domains.

LEMMA 4. *Let* H *be a* G*-domain and let* $a_1, \dots, a_n$ *be any given non-trivial elements of* H. *Then there exist elements* $g_2, \dots, g_n$ *in* G *such that*

$$[a_1, a_2^{g_2}, \dots, a_n^{g_n}] \neq 1.$$

Proof. Since H has no G-zero divisors, there exists an element $g_2 \in G$ such that $[a_1, a_2^{g_2}] \neq 1$. The same argument applies now to $[a_1, a_2^{g_2}]$ and a_3 and so the desired conclusion follows inductively.

There are many other G-domains besides these CSA-groups, as the following theorem shows.

THEOREM A1. *If U is a domain, if T is a torsion-free domain then the wreath product $U \wr T$ is a domain.*

Proof. In order to prove that $W = U2T$ is a domain, suppose that $x, y \in W$ are a pair of non-trivial elements. Now W is the semi-direct product of B and T, where B, the normal closure of U in W, is the direct product of the conjugates U^t of U by the elements $t \in T$. We have to find an element $z \in U \wr T$ such that $[x, y^z] \neq 1$. If $x, y \in B$, then we can find an element $t \in T$ such that the supports of x and y^t overlap. In view of the fact that U is a domain, we can find an element $g \in U$ such that $[x, y^{tg}] \neq 1$ and so we can take $z = tg \in U \wr T$. If x, y are non-trivial modulo B, then simply by going over to the quotient group W/B $(\simeq T)$, the existence of $z \in T$ and hence in $U \wr T$, is immediate. Finally, suppose that $y \in B$ and $x \notin B$. Now $x = tb$, where $t \in T$, $t \neq 1$, $b \in B$. We need to express y in the form

$$y = b_1 \cdots b_m,$$

where b_i is a non-trivial element in U^{t_i} and $t_1, \dots, t_m$ are distinct elements of T. We claim that x does not commute with y (and hence that we can choose $z = 1$). Otherwise

$$\{t_1 t, \dots, t_m t\} = \{t_1, \dots, t_m\}.$$

This means that right multiplication by t gives rise to a permutation of the finite set $\{t_1, \dots, t_m\}$. Consequently a big enough power of t induces the identity permutation on $\{t_1, \dots, t_m\}$ which implies that $t_1 t^k = t_1$ for a large enough choice of k. But T is torsion-free and so this is impossible. This completes the proof of the theorem.

We now assume the hypothesis and notation of the above theorem. Then we have the following simple proposition.

PROPOSITION 2. *Let $W = U \wr T$. Then the following hold*

1. *For an arbitrary finite set Z of elements from $U \wr T$ there exists a pair of non-trivial elements $x, y \in B$ such that $[x, y^z] = 1$ for each $z \in Z$;*
2. *W is not a CSA-group; indeed, it is not even commutative transitive.*

Proof. (1) Let B be the normal closure of U in W. Then B is the direct product of the conjugates U^t of U by the elements $t \in T$. If we fix a finite set Z, then $[x, y^z] = 1$ for each $z \in Z$, provided the supports of x and y^z are disjoint. This is easy to arrange since T is torsion-free.

(2) Since U is a domain, it is non-abelian. It follows that B is not commutative transitive and hence neither is W.

It follows that we can use wreath products to construct G-domains which are not CSA-groups.

We prove next the following theorem.

THEOREM A2. *Let A and B be domains. Suppose that C is a subgroup of both A and B satisfying the following condition*:

$$\text{if } c \in C,\ c \neq 1, \text{ either } [c, A] \not\subseteq C \text{ or } [c, B] \not\subseteq C. \qquad (*)$$

Then the amalgamated free product $H = A *_C B$ *is a domain.*

Proof. We make use of the usual notation and terminology for working with elements of amalgamated products, described, e.g., in [KMS]. Let us note that if $C = A$ or $C = B$ then H is one of the factors and we have nothing to prove. We can assume therefore that C is a proper subgroup of both factors. The condition $(*)$ implies that if $C \neq 1$ then C is not simultaneously normal in both factors; in particular, its index is greater than 2 in at least one of the factors. Notice also that since both A and B are domains, neither of them is abelian, unless they are trivial.

Suppose that x is a zero divisor in H. Then there exists a non-trivial element $y \in H$ such that $[y^h, x^{h'}] = 1$, for every choice of elements h, h' in H. So, in particular, $[x, y] = 1$. It follows then from a theorem of A. Karrass and D. Solitar (see [MKS, Theorem 4.5]) that this is impossible unless one of the following conditions holds:

1. Either x or y belongs to some conjugate of the amalgamated subgroup C;

2. Neither x nor y is in a conjugate of C, but x is in a conjugate of a factor (A or B) and y is in the same conjugate of that factor;

3. Neither x nor y is in a conjugate of a factor and $x = g^{-1}cgz^n$, $y = g^{-1}c^*gz^m$, where $c, c^* \in C$, and $g^{-1}cg$, $g^{-1}c^*g$, and z commute pairwise.

We consider these three cases in turn.

If $x \in C^g$ then we can assume that $x \in C$. If $y \in A \cup B$, then $[y, x^w] \neq 1$ for a suitable choice of $w \in A \cup B$, because A and B are domains. So $y \notin A \cup B$. Choose $h \in H$ so that $y' = y^h$ is cyclically reduced. By the remark above we can assume that y' is of length at least two. So, replacing y' by its inverse if necessary, we find that

$$y' = a_1 b_1 \cdots a_n b_n \qquad (n \geq 1),$$

where $a_i \in A - C$, $b_i \in B - C$, for each choice of i. As we mentioned above C is not normal in both A and B. Therefore there exists an element $v \in A \cup B$ such that $x^v \notin C$. For definiteness, suppose that $x^v \in A$. Then

$$x^v y' \neq y' x^v,$$

because $x^v y'$ is of length at most n, whereas $y' x^v$ is of length $n + 1$. Thus the first case cannot arise.

We consider next, the second case. We can assume here that both x and y belong to one and the same factor. Since each factor is a domain, there exists an element t in the appropriate factor such that $[y, x^t] \neq 1$, which means that x is not an H-zero divisor. Hence this case cannot arise.

We are left with the third possibility. We can assume here that $x = cz^n$, $y = c_1 z^m$, $c, c_1 \in C$, and c, c_1, z commute pairwise, and that the length of z is at least 2. We claim that there exists an element $f \in H$ such that if we write the elements z and f in reduced form then the products $cz^n f^{-1} cz^m f$ and $f^{-1} cz^m fcz^n$ are also in reduced form and not equal to one another. Indeed, if we choose the first syllable of f appropriately, then we can make sure that the product $f^{-1} cz^m f$ is reduced. Similarly, if we choose the last syllable of f appropriately, we can arrange that the products $cz^n f^{-1}$ and fcz^n are reduced and moreover, that the last syllables in f and z either lie in different factors or lie in the same factor but have different right coset representatives. It follows then that $[y, x^{f^{-1}}] \neq 1$. This completes the proof of the theorem.

We mention here two consequences of this theorem.

Corollary 2. *Let A and B be domains. Then $A * B$ is a domain.*

Corollary 3. *Let A and B be domains and let C be a subgroup of both of them. If C is malnormal either in A or in B, then $A *_C B$ is a domain.*

It is worth pointing out that the condition $(*)$, above, is essential.

Example 1. *Let F be free on x and y and $\overline{F}$ free on $\bar{x}$ and $\bar{y}$. Let, furthermore, $C = \mathrm{gp}_F(x)$ and $\overline{C} = \mathrm{gp}_{\overline{F}}(\bar{x})$. Then $H = F *_{C=\overline{C}} \overline{F}$ is not an H-domain.*

Proof. The elements y and $\bar{y}$ have exactly the same action (by conjugation) on the normal subgroup C in the group H. Hence the element $y^{-1}\bar{y}$ centralizes the subgroup C. It follows that $[y^{-1}\bar{y}, \mathrm{gp}_H(x)] = 1$; i.e., x is an H-zero divisor.

Finally we have

THEOREM A3. *The free product, in the category of G-groups, of two G-domains A and B is a G-domain whenever G is malnormal in both A and B.*

The proof can be carried out along the same lines as the proof of the theorem above. We note only that the first two cases are completely analogous, while the last case is even easier. It suffices here to note only that a non-trivial element of G cannot commute with an element of either A or B which is not contained in G.

It follows immediately then from this theorem that if A and B are free and G is a maximal cyclic subgroup of both of them, then the free product of A and B in the category of G-groups is a G-domain.

2.2. *Equationally Noetherian Groups*

We have already defined what it means for a G-group H to be G-equationally Noetherian. This notion is very different from what is known as the Noetherian condition in everyday group theory. Recall that a group is said to be Noetherian if it satisfies the maximum condition on its subgroups. Another condition that has turned out to be useful is the maximal condition on normal subgroups introduced by Philip Hall (see [BR]). However neither of these chain conditions turn out to be useful in the context of algebraic geometry over groups. We recall the relevant definition that we introduced in Section 1.3.

DEFINITION 4. A G-group H is said to be G-equationally Noetherian if for every $n > 0$ and every subset S of $G[x_1, \dots, x_n]$ there exists a finite subset S_0 of S such that

$$V(S) = V(S_0).$$

In the case $H = G$ we omit all mention of G and simply say that *H is equationally Noetherian* group.

Sometimes, if $S, S_0 \subseteq G[x_1, \dots, x_n]$ and $V_H(S) = V_H(S_0)$ we say that *the systems of equations $S = 1$ and $S_0 = 1$ are equivalent over H*. If $G = 1$, then a subset S of $G[X]$ is termed *coefficient-free* and we refer to the system $S = 1$ of equations as a *coefficient-free system*.

We, respectively, denote by $\mathscr{E}$ and $\mathscr{E}_G$ the class of all equationally Noetherian groups and the class of all G-equationally Noetherian groups. In particular, $\mathscr{E}_1$ is the class of all G-equationally Noetherian groups with $G = 1$ (i.e., the class of all groups that satisfy the Noetherian condition with respect to coefficient-free systems of equations). If $G' \leq G$ then every G-group H can be viewed also as a G'-group; clearly $\mathscr{E}_G \subseteq \mathscr{E}_{G'}$. It follows that an equationally Noetherian group H is G-equationally Noetherian for every choice of the subgroup G of H. The converse, properly formulated, also holds.

PROPOSITION 3. *Let H be a G-group. If G is finitely generated and $H \in \mathscr{E}_1$, then $H \in \mathscr{E}_G$.*

Proof. Let S be a subset of $G[x_1, \dots, x_n]$. Each element $w \in S$ can be expressed functionally in the form

$$w = w(\bar{x}, \bar{a}) = w(x_1, \dots, x_n, a_1, \dots, a_k),$$

where $a_1, \dots, a_k$ is a finite generating set for G. Let us replace the generators a_i by new variables, say y_i. The new set $S(\bar{x}, \bar{y}) \subseteq G[\bar{x}, \bar{y}]$ is coefficient-free; hence $V_H(S(\bar{x}, \bar{y})) = V_H(S_0(\bar{x}, \bar{y}))$ for some finite subset $S_0(\bar{x}, \bar{y}) \subseteq S(\bar{x}, \bar{y})$. Now consider the set of solutions of the system $S_0(\bar{x}, \bar{y}) = 1$ for which $y_i = a_i$. This is exactly the algebraic set $V_H(S_0(\bar{x}, \bar{a}))$. But these solutions also satisfy the whole system $S(\bar{x}, \bar{y}) = 1$; therefore $V_H(S_0(\bar{x}, \bar{a})) = V_H(S(\bar{x}, \bar{a})) = V_H(S)$.

We have been unable to resolve the following problem.

Problem 1. Let H be a G-group such that every one-variable system S contained in $G[x]$ is equivalent over H to a finite subset of itself. Does this imply that H is G-equationally Noetherian?

The following theorem is, in a sense, due to Bryant ([BR], 1977), in the one variable case; it was reproved by Guba ([GV], 1986), in the case of free groups. Both proofs of the authors cited are similar and can be carried over to a proof of the following theorem.

THEOREM B1. *Let H be a linear group over a commutative, Noetherian, unitary ring, e.g., a field. Then H is equationally Noetherian.*

Proof. Consider first the case where H is a subgroup of the general linear group $GL(n, K)$ over a field K. We can think of matrices from $GL(n, K)$ as elements of the K-vector space K^{n^2} of dimension n^2. The classical Zariski topology on K^{n^2}, which is Noetherian by Hilbert's basis theorem, induces the usual Zariski topology on $GL(n, K)$ and, consequently also on H, which is therefore Noetherian in this topology. Since multiplication and inversion in H are continuous functions in this induced Zariski topology, for every element $w(x_1, \dots, x_n) \in G[x_1, \dots, x_n]$, the set of all roots $V_H(w)$ of w is closed in this topology, since it is the pre-image of 1 under the continuous map $H^n \to H$ defined by w. This implies that every algebraic set over H is closed in this topology, because such sets are exactly intersections of sets of roots of single elements in $G[x_1, \dots, x_n]$. In other words, the classical Zariski topology on H is a refinement of the non-commutative analogue of the Zariski topology on H that we have introduced here. It follows that this non-commutative Zariski topology is also Noetherian, hence H is equationally Noetherian (see Theorem D1).

In general, the group $GL(n, R)$ over a commutative, Noetherian, unitary ring R is a subgroup of a direct product of finitely many linear groups over a field (see [WB]); hence it is equationally Noetherian by Theorem B2 below.

As we noted earlier, not all equationally Noetherian groups are linear; we prove, below, that all abelian groups are equationally Noetherian, thereby providing some additional examples (see [WB]). There are other examples of finitely generated equationally Noetherian non-linear groups, which are due to Bryant. He proved [BR] that finitely generated abelian-by-nilpotent groups are equationally Noetherian. Since the wreath product of a non-trivial, finitely generated abelian group U by a finitely generated nilpotent group T is linear if and only if T is virtually abelian [WB], this provides us with more equationally Noetherian groups that are not linear. The paper [BMRO] contains a further discussion about equationally Noetherian groups as well as additional examples of various kinds.

We prove now the following theorem.

THEOREM 1. *Every abelian group A is equationally Noetherian.*

Proof. Every system $S = 1$ of equations over an abelian group A is equivalent to a linear system, obtained by abelianizing each of the elements in S. Each such linear system over A can be re-expressed in the form

$$S : m_{i1}x_1 + \cdots m_{in}x_n = a_i \qquad \big(i \in I, m_{ij} \in Z, a_i \in A\big).$$

As usual (applying the Euclidean algorithm), this system is equivalent to a finite linear system $f_i = b_i$ ($b_i \in A$), $i = 1, \dots, k$, in row-echelon form. So $k \leq n$. Notice, that all the equations $f_i = b_i$ are some integer linear combinations of finite family S_0 of equations from the original system $S = 1$. Hence, $V_A(S_0) = V_A(S)$. This completes the proof.

It is important to notice that the system of equations in row-echelon form obtained above consists of no more than n equations. However S_0 can contain an arbitrarily large number of equations (see the example below).

EXAMPLE 2. *Let $p_1, \dots, p_n$ be distinct primes and suppose that a group H has elements of orders $p_1, \dots, p_n$. Then the system*

$$x^{p_1 \cdots p_{i-1} p_{i+1} \cdots p_n} = 1, \qquad i = 1, \dots, n,$$

is equivalent to the system

$$x = 1,$$

which is in row-echelon form. If we now re-express this latter system in terms of the original one, we find that all of the original equations are needed.

Examples of nilpotent groups which are not G-equationally Noetherian are plentiful—see e.g., Proposition 4, 1 below. So Theorem 1 cannot be generalized to nilpotent groups.

The example above suggests the following

Problem 2. Let H be a G-group. Suppose that for every integer $n > 0$ and every subset S of $G[x_1, \dots, x_n]$ there exists a finite subset S_0 of $G[x_1, \dots, x_n]$ such that $V_H(S) = V_H(S_0)$. Is H G-equationally Noetherian?

We are now in a position to formulate our next theorem.

THEOREM B2. *Let $\mathscr{E}_G$ be the class of all G-equationally Noetherian G-groups. Then the following hold*:

1. *$\mathscr{E}_G$ is closed under G-subgroups, finite direct products and ultra-powers*;

2. *$\mathscr{E}_G$ is closed under G-universal (G-existential) equivalence, i.e., if $H \in \mathscr{E}_G$ and H' is G-universally (G-existentially) equivalent to H, then $H' \in \mathscr{E}_G$*;

3. *$\mathscr{E}_G$ is closed under separation*; *i.e., if $H \in \mathscr{E}_G$ and H' is G-separated by H, then $H' \in \mathscr{E}_G$.*

Proof. 1. Suppose, first, that a G-group H is G-equationally Noetherian and that H' is a G-subgroup of H. Let S be a subset of $G[x_1, \dots, x_n]$. Since H is G-equationally Noetherian, there is a finite subset S_0 of S such that $V_H(S) = V_H(S_0)$. We claim that $V_{H'}(S) = V_{H'}(S_0)$. Indeed, if $v \in V_{H'}(S_0)$, then $v \in V_H(S_0) = V_H(S)$. Hence $v \in V_{H'}(S)$. It follows that $V_{H'}(S) = V_{H'}(S_0)$, as desired.

Suppose next that $H_1, \dots, H_k$ are G-equationally Noetherian G-groups and that D is their direct product. According to the remarks made in the Introduction, we can turn D into a G-group by choosing the copy of G in D to be the diagonal subgroup $\{(g, \dots, g) | g \in G\}$ of D. Now let S be a subset of $G[x_1, \dots, x_n]$. Then for each i, $V_{H_i}(S) = V_{H_i}(S_i)$, where S_i is a finite subset of S. Put $S_0 = S_1 \cup \cdots \cup S_k$. Then $V_D(S) = V_D(S_0)$. This completes the proof of the most of 1. The statement about ultrapowers follows from 2. as indicated below.

2. Suppose that $S \subseteq G[x_1, \ldots, x_n]$. Now $V_H(S) = V_H(S_0)$, where $S_0 = \{f_1, \ldots, f_k\}$ is a finite subset of S. For each $f \in S$, let

$$\phi_f = \forall x_1 \forall x_2 \cdots \forall x_n \left(\bigwedge_{i=1}^{k} f_i(x_1, \ldots, x_n) = 1 \to f(x_1, \ldots, x_n) = 1 \right).$$

Let Φ be the set of all such sentences. Then all of the sentences in Φ are satisfied in H and hence also in H'. This translates into $V_{H'}(S) = V_{H'}(S_0)$, as desired.

Notice that an ultrapower U of a G-group H is G-elementarily equivalent to H (see, e.g., [CK]). So, in particular, U and H are G-universally equivalent. Consequently, if $H \in \mathcal{E}_G$, then $U \in \mathcal{E}_G$.

3. Let S be a subset of $G[X]$. Then there exists a finite subset S_0 of S such that $V_H(S) = V_H(S_0)$. We claim that this implies that $V_{H'}(S) = V_{H'}(S_0)$. For if this is not the case, there exists $v = (a_1, \ldots, a_n) \in H'^n$ such that S_0 vanishes at v but S does not vanish at v. So there exists an element $f \in S$ such that $f(v) \neq 1$. Choose now a G-homomorphism ϕ of H' into H so that $\phi(f(v)) = f(\phi(a_1), \ldots, \phi(a_n)) \neq 1$. But for each $f' \in S_0$, $f'(v) = 1$ and therefore $\phi(f'(v)) = f'(\phi(a_1), \ldots, \phi(a_n)) = 1$; i.e., $(\phi(a_1), \ldots, \phi(a_n))$ is a root of S_0. So $(\phi(a_1), \ldots, \phi(a_n))$ is a root of S and therefore $f(\phi(a_1), \ldots, \phi(a_n)) = 1$, a contradiction. This completes the proof of Theorem B2.

In general the restricted direct product of G-equationally Noetherian groups need not be G-equationally Noetherian. On the other hand, direct powers of groups from $\mathcal{E}_1$ still belong to $\mathcal{E}_1$. This is the content of the following proposition.

PROPOSITION 4. 1. *Let $\{H_i | i \in I\}$ be a family of G_i-equationally Noetherian G_i-groups. If there are infinitely many indices i for which G_i is not in the center of H_i, then the restricted direct product $D = \prod_{i \in I} H_i$ of the groups H_i, viewed as a $G = \prod_{i \in I} G_i$-group, is not G-equationally Noetherian.*

2. *The class $\mathcal{E}_1$ is closed under unrestricted and restricted direct powers.*

Proof. 1. We choose an infinite subset J of I and elements $a_j \in G_j$ which are not in the center of H_j, $j \in J$. Consider now the subset $S = \{[x, a_j] \| j \in J\}$ of $G[x]$. Then S is not equivalent to any of its finite subsets.

2. Let $H \in \mathscr{E}_1$ and let I be a set of indices. Denote by H^I the unrestricted Ith power of H. Let $S \subseteq F(X)$, where $F(X)$ is the free group on X (i.e., $S = 1$ is a coefficient-free system of equations). Then $V_H(S) = V_H(S_0)$ for some finite subset S_0 of S. It is easy to see that $V_{H^I}(S) = V_{H^I}(S_0)$.

In view of the fact that $\mathscr{E}_1$ is closed under subgroups, this suffices for the proof of the proposition.

The next result provides a description of the Baumslag–Solitar groups which are equationally Noetherian.

PROPOSITION 5. *Let*

$$B_{m,n} = \langle a, t; t^{-1}a^m t = a^n \rangle \qquad (m > 0, n > 0).$$

Then $B_{m,n}$ is equationally Noetherian provided either $m = 1$ or $n = 1$ or $m = n$; in all other cases $B_{m,n}$ does not belong to $\mathscr{E}_1$.

Proof. If either $m = 1$ or $n = 1$, then $B_{m,n}$ is metabelian and linear. If $m = n$, observe that the normal closure N of t and a^m is the direct product of a free group of rank m and the infinite cyclic group on a^m. Moreover N is of index m in $B_{m,m}$. Now N is linear and hence so too is $B_{m,m}$. Thus in all the cases above $B_{m,n}$ is equationally Noetherian.

Suppose then that $m \neq 1$, $n \neq 1$, and $m \neq n$. On replacing t by t^{-1} if necessary, we can assume that m does not divide n. Observe that the elements

$$a^{m^k}, t^{-1}a^{m^k}t, \dots, t^{-k}a^{m^k}t^k, \tag{6}$$

all commute, but if j is chosen sufficiently large, a does not commute with $t^{-j}a^{m^k}t^j$. Let

$$S = \left\{ \left[x_1, x_2^{-i}x_1x_2^i \right] | i = 1, 2, \dots \right\} \subseteq F(x_1, x_2).$$

Suppose that $B_{m,n} \in \mathscr{E}_1$. Then there exists an integer $l > 0$ such that

$$V(S) = V\left(\left\{ \left[x_1, x_2^{-i}x_1x_2^i \right] | i = 1, 2, \dots, l \right\} \right).$$

But this implies that

$$(a^{m^l}, t^{-j}a^{m^l}t^j) \in V(S),$$

contradicting (6).

Notice that in the case of commutative rings, if R is a Noetherian ring then the ring of polynomials $R[x]$ is also Noetherian. We do not know whether the corresponding result, which we formulate here as a conjecture, also holds.

Conjecture 1. Let G be an equationally Noetherian group. Then the free G-group $G[x_1, \dots, x_n]$ is G-equationally Noetherian.

We can prove this conjecture for various classes of equationally Noetherian groups, by making use of the specific properties of the groups involved. However, the general conjecture remains unsolved. The best we can do is to prove the following theorem.

THEOREM 2. *Let G be a linear group or a torsion-free hyperbolic equationally Noetherian group. Then the group $G[x_1, \dots, x_n]$ is also equationally Noetherian.*

Proof. Suppose, first, that G is a linear group. The free product of G and a free group $F(x_1, \dots, x_n)$ is again linear (see, for example, [WB, p. 35]); hence $G[x_1, \dots, x_n]$ is equationally Noetherian.

If G is a torsion-free hyperbolic group, then the group $G[X]$ is G-separated by the G-equationally Noetherian group G [BMR2]. Consequently $G[X]$ is G-equationally Noetherian by Theorem B2. Since $G[x_1, \dots, x_n]$ is finitely generated and in $\mathscr{E}_1$, then by Proposition 3 $G[x_1, \dots, x_n]$ is equationally Noetherian.

The most general open questions in this direction are

Problem 3. Is the free product of two equationally Noetherian groups equationally Noetherian?

Problem 4. Is an arbitrary hyperbolic group equationally Noetherian?

2.3. *Separation and Discrimination*

In this section we prove some results for groups; the corresponding result hold also for rings, but we restrict our attention here only to groups.

We recall first some definitions given in the Introduction.

DEFINITION 5. Let H be a G-group. Then we say that a family

$$\mathscr{D} = \{D_i | i \in I\},$$

of G-groups G-separates H if for each non-trivial $h \in H$ there exists a group $D_i \in \mathscr{D}$ and a G-homomorphism $\phi\colon H \to D_i$ such that $\phi(h) \neq 1$.

We say that $\mathscr{D}$ G-discriminates H if for each finite subset $\{h_1, \dots, h_n\}$ of non-trivial elements of H there exists a $D_i \in \mathscr{D}$ and a G-homomorphism $\phi\colon H \to D_i$ such that $\phi(h_j) \neq 1$, $j = 1, \dots, n$.

If $\mathscr{D}$ consists of the singleton D, then we say that D G-separates H in the first instance and that D G-discriminates H in the second. If G is the trivial group, then we omit any mention of G and simply say that $\mathscr{D}$ separates H or $\mathscr{D}$ discriminates H. These notions of separation and discrimination are often expressed in the group-theoretical literature by saying, respectively, that H is *residually* $\mathscr{D}$ and that H is *fully residually* $\mathscr{D}$ or ω*-residually* $\mathscr{D}$.

DEFINITION 6. Let $\mathscr{K}_G$ be the category of all G-groups, let $\mathscr{S}_G \subset \mathscr{K}_G$ be the subcategory of all G-groups G-separated by the singleton G and let $\mathscr{D}_G$ be the subcategory of $\mathscr{S}_G$ consisting of those G-groups which are G-discriminated by G.

If the group G has no G-zero divisors then according to [BMR2] there exists a very simple criterion for a group in $\mathscr{S}_G$ to belong to $\mathscr{D}_G$. B. Baumslag was the first to exploit this kind of argument in the case of free groups [BB].

THEOREM C1 [BMR2]. *Let G be a domain. Then a G-group H is G-discriminated by G if and only if H is a G-domain and H is G-separated by G.*

Proof. Let $H \in \mathscr{S}_G$ and suppose that H is a G-domain. Then for an arbitrary finite set $h_1, \dots, h_n$ of non-trivial elements of H there exist elements $z_2, \dots, z_n \in H$ such that the left-normed commutator $c = [h_1, h_2^{z_2}, \dots, h_n^{z^n}]$ is non-trivial. Hence we can separate c in G by a G-homomorphism ϕ: $H \to G$ such that $\phi(c) \neq 1$. But this implies that $\phi(h_i) \neq 1$ for all i. This shows that G G-discriminates H.

Suppose now that $H \in \mathscr{D}_G$ and that f, h are two nontrivial elements of H. Then there exists a G-homomorphism ϕ: $H \to G$ such that $\phi(f)$ and $\phi(h)$ are both nontrivial in G. Since G is a domain, it follows that $[\phi(f), \phi(h)^g] \neq 1$ for some $g \in G$. But then $[f, h^g] \neq 1$ in H, which shows that H is a G-domain.

It is not hard to see that if F is a non-abelian free group, then $F \times F$ is separated by F, but it is not discriminated by F—this remark is due to [BB].

Now we are given an important characterization of finitely generated G-groups which are G-universally equivalent to G, when G is equationally Noetherian. This characterization goes back to [RV1] in the case of free groups.

We say that the G-group H is *locally G-discriminated* by the G-group H' if every finitely generated G-subgroup of H is G-discriminated by H'.

THEOREM C2. *Let H and H' be G-groups and suppose that at least one of them is G-equationally Noetherian. Then H is G-universally equivalent to H'*

if and only if H is locally G-discriminated by H' and H' is locally G-discriminated by H.

Proof. Suppose that H is G-universally equivalent to H' and, furthermore, that one of them is G-equationally Noetherian. By Theorem B2 both H and H' are G-equationally Noetherian.

Let K be a finitely generated G-subgroup of H and let

$$K = \langle x_1, \ldots, x_n; r_1, r_2, \ldots, \rangle$$

be a G-presentation of K with finitely many G-generators. The system of equations $\{r_i(x_1, \ldots, x_n) = 1 | i = 1, 2, \ldots\}$ is equivalent over H' to one of its finite subsets, say, $\{r_i(x_1, \ldots, x_n) = 1 | i \leq m\}$. Let $u_1(x_1, \ldots, x_n), \ldots, u_k(x_1, \ldots, x_n)$ be arbitrary non-trivial elements from K. We have to find a homomorphism from K to H' which separates the given elements $u_1(x_1, \ldots, x_n), \ldots, u_k(x_1, \ldots, x_n)$ from the identity. The formula

$$\Phi = \exists x_1 \cdots x_n \left(\bigwedge_1^m r_i(x_1, \ldots, x_n) = 1 \bigwedge_1^k u_j(x_1, \ldots, x_n) \neq 1 \right)$$

holds in H and thus it holds in H'. Consequently, there exist elements $h_1, \ldots, h_n$ in H' such that

$$r_1(h_1, \ldots, h_n) = 1, \ldots, r_m(h_1, \ldots, h_n) = 1,$$

but

$$u_1(h_1, \ldots, h_n) \neq 1, \ldots, u_k(h_1, \ldots, h_n) \neq 1.$$

It follows that $r_i(h_1, \ldots, h_n) = 1$, $i = 1, 2, \ldots$. Hence, the map

$$x_1 \to h_1, \ldots, x_n \to h_n$$

can be extended to a G-homomorphism ϕ: $K \to H$ which separates the elements $u_1(x_1, \ldots, x_n), \ldots, u_k(x_1, \ldots, x_n)$. This proves that K is G-discriminated by H'. Since K was an arbitrarily finitely generated G-subgroup of H it follows that H is locally G-discriminated by H'. Similarly, H' is locally G-discriminated by H.

Now suppose that the G-group H is locally G-discriminated by the G-group H'. We claim that any $\exists$-sentence with constants from G which holds in H holds also in H'. Let us consider a formula of the type

$$\Phi = \exists x_1, \ldots, x_n \left(\bigwedge_1^s u_i(x_1, \ldots, x_n) = 1 \bigwedge_1^t v_j(x_1, \ldots, x_n) \neq 1 \right),$$

where the words u_i and v_j may contain constants from G. Let the elements $a_1, \ldots, a_n \in H$ satisfy this sentence in H. Denote by K the G-subgroup generated by $\{a_1, \ldots, a_n\}$. By hypothesis, there exists a G-homomorphism $f\colon K \to H'$ which separates the elements

$$v_1(a_1, \ldots, a_n), \ldots, v_t(a_1, \ldots, a_n),$$

in H'. This implies that the images $f(a_1), \ldots, f(a_n)$ satisfy the same equalities $u_i(f(a_1), \ldots, f(a_n)) = 1$, $i = 1, \ldots, s$, and the same inequalities

$$v_i(f(a_1), \ldots, f(a_n)) \neq 1, \qquad i = 1, \ldots, t,$$

in H'. Therefore, the sentence Φ holds in H'. This shows that H is G-existentially equivalent to H', and hence H is G-universally equivalent to H'.

A G-subgroup H of the G-group H' is said to be *G-existentially closed* in H', if any existential sentence with constants from G holds in H' if and only if it holds in the subgroup H.

COROLLARY 4. *Let H be a G-group which is locally G-discriminated by G. Then G is G existentially closed in H and, in particular, G is universally equivalent to H.*

The proof follows along exactly the same lines as the proof of the second part of Theorem C2.

2.4. *Universal Groups*

The discussion in this subsection stems from a consideration of a well-known fact from the field theory. If K is a given field and $\overline{K}$ is its algebraic closure then any finite algebraic extension of K is embeddable in $\overline{K}$; i.e., $\overline{K}$ is *universal for fields which are finitely generated* (*as modules*) *over* K. This remark leads to

DEFINITION 7. Let $\mathscr{K}$ be a category of groups and let λ be a cardinal number. We term a group $H \in \mathscr{K}$ λ-universal in $\mathscr{K}$ if every group in $\mathscr{K}$ generated by fewer than λ generators is embeddable in H and conversely, every subgroup of H generated by fewer than λ generators belongs to $\mathscr{K}$.

The main goal of this subsection is to discuss universal groups with respect to the categories $\mathscr{S}_G$ and $\mathscr{D}_G$. In this case all the notions from the definition above are relative to the category of G-groups; i.e., all embeddings are G-monomorphisms and all the subgroups in consideration are G-subgroups. We focus mostly on the case of $\lambda = \aleph_0$, but most of the results can be carried over to the general case.

Suppose now that the G-group H is separated by the family $\mathscr{D} = \{D_i | i \in I\}$ of G-groups. For each $h \in H$, $h \neq 1$, there exists an index $i \in I$ and a G-homomorphism θ_i: $H \to D_i$ such that $\theta_i(h) \neq 1$. Let $K_{i,h}$ be the kernel of θ_i. Then $K_{i,h}$ is an ideal of H and

$$\bigcap_{\substack{h \in H \\ h \neq 1}} K_{i,h} = 1.$$

It follows that the canonical homomorphism

$$\theta \colon H \to P = \prod_{\substack{h \in H \\ h \neq 1}} H/K_{i,h} \qquad (7)$$

is an injective G-homomorphism of the G-group H into the G-group P, where in this instance we take the diagonal of all the copies of G in the various factors to be the designated copy of G in P. Notice that $H/K_{i,h}$ is G-isomorphic to a G-subgroup of D_i and hence H is G-isomorphic to a G-subgroup of the G-group obtained by forming the unrestricted direct product of all of the groups in $\mathscr{D}$.

If $\mathscr{D}$ is just the singleton $\{G\}$, then it is not hard to deduce the following proposition.

PROPOSITION 6. *Let I be a set of indices and suppose its cardinality is not less than the maximum of the cardinalities of G and $\aleph_0$. Then the unrestricted direct power (viewed as a G-group)*

$$G_{\aleph_0}(I) = \prod_{i \in I} G_i \qquad (G_i \simeq G)$$

is an $\aleph_0$-universal group in the category $\mathscr{S}_G$.

Proof. The group $G_{\aleph_0}(I)$ lies in $\mathscr{S}_G$ since every non-trivial element $h \in G_{\aleph_0}(I)$ has a non-trivial coordinate h_i for some $i \in I$. Hence the canonical projection into the ith factor $G_{\aleph_0}(I) \to G$ separates h in G. Moreover, for the same reason, every G-subgroup of $G_{\aleph_0}(I)$ also lies in $\mathscr{S}_G$. On the other hand, every finitely generated G-group from $\mathscr{S}_G$ is embeddable in $G_{\aleph_0}(I)$. Indeed, the cardinality of any finitely generated G-group is not greater than the maximum of the cardinality of G and $\aleph_0$. Now the result follows from the (7).

The G-subgroups of $G_{\aleph_0}(I)$ which are G-discriminated by G can be characterized by considering the supports of their elements. In order to explain, let $I(g)$ be the set of all indices $i \in I$ such that the ith coordinate of the element $g \in G_{\aleph_0}(I)$ is non-trivial. Now if H is a G-subgroup of $G_{\aleph_0}(I)$, put

$$I(H) = \{I(h) | h \in H, h \neq 1\}.$$

PROPOSITION 7. *Let G be a domain. Then a G-subgroup H of $G_{\aleph_0}(I)$ is G-discriminated by G if and only if the set $I(H)$ satisfies the following condition: the intersection of an arbitrary finite collection of sets from $I(H)$ is non-empty.*

Proof. Let $h_1, \dots, h_n$ be non-trivial elements of the G-subgroup H of $G_{\aleph_0}(I)$ which is G-discriminated by G. By Theorem C1, H is a G-domain. Hence there are elements $z_2, \dots, z_n \in H$ such that the commutator

$$c = [h_1, h_2^{z_2}, \dots, h_n^{z_n}]$$

is non-trivial. Consequently, $I(c)$ is non-empty and

$$I(c) \subseteq I(h_1) \cap \cdots \cap I(h_n).$$

Therefore $I(h_1) \cap \cdots \cap (h_n)$ is non-empty.

On the other hand, suppose that the G-subgroup H of $G_{\aleph_0}(I)$ satisfies the given condition. Let $h_1, \dots, h_n$ be non-trivial elements of H. Then $I(h_1) \cap \cdots \cap I(h_n)$ is non-empty. Let $i \in I$ belong to this intersection. Then the projection of $G_{\aleph_0}(I)$ into the ith factor G is a G-homomorphism of H onto G which maps each of the h_i into non-trivial elements of G, as required.

The following result is a corollary of the proof above rather than the proposition itself.

COROLLARY 5. *Let H be a G-domain. Suppose H is a G-subgroup of the direct product of the finitely many G-groups $H_1, \dots, H_n$. If we denote by λ_i the restriction to H of the canonical projection $H_1 \times \cdots \times H_n \to H_i$, then at least one of $\lambda_1, \dots, \lambda_n$ is an embedding.*

Proof. Suppose that for each $i = 1, \dots, n$ there exists a non-trivial element, say c_i, in the kernel of the homomorphism λ_i. Then the element $c = [c_1, c_2^{g_2}, \dots, c_n^{g_n}]$ is also non-trivial for some choice of the elements $g_2, \dots, g_n \in G$. It follows that $\lambda_i(c) = 1$ for each $i = 1, \dots, n$, which is impossible since $c \neq 1$. This completes the proof.

Now we are able to describe some groups which are "almost" $\aleph_0$-universal in the category $\mathcal{D}_G$. Let I be an arbitrary countable set of indices. A set U of non-empty subsets of I is called an *ultrafilter over* I if the following conditions hold:

1. U is closed under finite intersections;
2. If $X \in U$ and Y is any subset of I containing X, then $Y \in U$;
3. For every subset X of I of either $X \in U$ or $I - X \cap U$.

The ultrafilter U over I gives rise to a normal subgroup N of the Cartesian product $G_{\aleph_0}(I)$, which is defined as

$$N = \{ g \in G_{\aleph_0}(I) | I - I(g) \in U \}.$$

The quotient group of $G_{\aleph_0}(I)$ by N is called the *ultrapower* of G with respect to the ultrafilter U, we denote this group by $G_{\aleph_0}(I)/U$. An ultrafilter U is said to be *non-principal* if there is no element $i \in I$ such that $i \in X$ for every $X \in U$.

PROPOSITION 8. *Let G be an equationally Noetherian group. Then for any countably infinite set I and any non-principal ultrafilter U over I, the ultrapower $G_{\aleph_0}(I)/U$ has the following properties*:

1. *$G_{\aleph_0}(I)/U$ is a G-group, where G is embedded in $G_{\aleph_0}(I)/U$ via the diagonal mapping*;
2. *Every finitely generated G-group in $\mathscr{D}_G$ is G-embeddable in $G_{\aleph_0}(I)/U$*;
3. *Every finitely generated G-subgroup of $G_{\aleph_0}(I)/U$ belongs to $\mathscr{D}_G$.*

Proof. By Theorem C2 every finitely generated G-group H in $\mathscr{D}_G$ is universally equivalent to G, hence by standard arguments from logic (see, for example, [CK]), H is embeddable in $G_{\aleph_0}(I)/U$. On the other hand all finitely generated G-subgroups of $G_{\aleph_0}(I)/U$ are universally equivalent to G (again see the book [CK]) and hence by Theorem C2 they belong to $\mathscr{D}_G$.

Notice, that in general the ultrapower $G_{\aleph_0}(I)/U$ does not belong to the category $\mathscr{D}_G$ therefore it is not a universal object for this category. Moreover, if $G \neq 1$ then the universal group $G_{\aleph_0}(I)$, as well as the group $G_{\aleph_0}(I)/U$, is uncountable whenever U is not a principal ultrafilter. The question as to whether there exist natural countable universal groups in the categories $\mathscr{S}_G$ and $\mathscr{D}_G$ is important for the further development of algebraic geometry over groups. In the case when G is a free non-abelian group Kharlampovich and Myasnikov proved [KM] that exponential group $F^{Z[x]}$ is an $\aleph_0$-universal group in the category $\mathscr{D}_G$. This result leads to

Conjecture 2. Let G be a non-abelian torsion-free hyperbolic group. Then the $Z[x]$-completion $G^{Z[x]}$ of G (see [MR2] for details) is an $\aleph_0$-universal group in the category $\mathscr{D}_G$.

Notice, that the group $G^{Z[x]}$ is G-discriminated by G [BMR2]. It follows therefore that in order to prove Conjecture 2 it suffices only to prove that any finitely generated G-group which is G-discriminated by G is G-embeddable into $G^{Z[x]}$.

3. AFFINE ALGEBRAIC SETS

3.1. *Elementary Properties of Algebraic Sets and the Zariski Topology*

Let G be a fixed group, let H be a G-group, and let n be a positive integer. We take for granted here the definitions and notation described in the Introduction. In view of its importance, however, we recall the following definition.

DEFINITION 8. Let S be a subset of $G[X]$. Then the set

$$V_H(S) = \{v \in H^n \mid f(v) = 1 \text{ for all } f \in S\}$$

is termed the (affine) algebraic set over H defined by S.

We sometimes denote $V_H(S)$ simply by $V(S)$. The following example provides us with algebraic sets.

EXAMPLE 3. *Let H be a G-group. Then the following subsets of H are algebraic*:

1. *The G-singleton $\{a\}$ (here $a \in G$),*

$$V_H(xa^{-1}) = \{a\};$$

2. *For any subset M of H, the centralizer $C_H(M)$*:

$$V_H(\{[x,m] \mid m \in M\}) = C_H(M).$$

The next lemma allows one to construct algebraic sets in "higher dimensions."

LEMMA 5. *Let H be a G-group and let U and W be affine algebraic sets in H^n and H^p, respectively. Then $U \times W$ is an algebraic set in H^{n+p}.*

The proof is analogous to that of the corresponding theorem in algebraic geometry. Indeed, if $U = V(S)$, where $S \subseteq G[x_1, \dots, x_n]$ and $W = V(T)$, where $T \subseteq G[y_1, \dots, y_p]$, then

$$U \times W = V(S \cup T),$$

where we view $S \cup T$ as a subset of $G[x_1, \dots, x_n, y_1, \dots, y_p]$.

The following lemma is useful.

LEMMA 6. *Let H be a G-group and H^n affine n-space over H. Then for arbitrary subsets S_i of $G[X]$, the following hold*:

1. $V_H(1) = H^n$;
2. $V_H(g) = \emptyset$ *for any non-trivial* $g \in G$;
3. $S_1 \subseteq S_2 \;\Rightarrow\; V_H(S_1) \supseteq V_H(S_2)$;

4. $V_H(S) = V_H(\mathrm{gp}_{G[X]}(S))$;
5. $\bigcap_{i \in I} V_H(S_i) = V_H(\bigcup_{i \in I} S_i)$;
6. *If H is a G-domain, then*

$$V_H(S_1) \cup V_H(S_2) = V_H\left([s_1, s_2^g] | s_i \in S_i, g \in G\right) = V_H\left(s_1 \diamond s_2 | s_i \in S_i, i = 1, 2\right);$$

7. *If H is a G-domain, then*

$$V_H(Q_1) \cup V_H(Q_2) = V_H(Q_1 \cap Q_2),$$

for any ideals Q_1, Q_2 of $G[X]$.

Proof. The verification of the first five assertions is straightforward and is left to the reader. In order to prove 6 we first prove that

$$V_H\left(\{[s_1, s_2^g] | s_i \in S_i, g \in G\}\right) \subseteq V_H(S_1) \cup V_H(S_2).$$

Suppose, that $v = (a_1, \dots, a_n) \in H^n$, $v \notin V_H(S_1) \cup V_H(S_2)$. So $v \notin V_H(S_1)$, $v \notin V_H(S_2)$. Hence there exist $s_1 \in S_1$ and $s_2 \in S_2$ such that

$$s_1(v) \neq 1, \qquad s_2(v) \neq 1.$$

Since H is a G-domain, there exists $g \in G$, such that

$$\left[s_1(v), s_2(v)^g\right] \neq 1.$$

Hence

$$v \notin V_H\left(\{[s_1, s_2^g] | s_i \in S_i, g \in G\}\right).$$

The reverse inclusion is immediate, which proves 6.

Now 7 follows from 6 since

$$Q_1 \diamond Q_2 \subseteq [Q_1, Q_2] \subset Q_1 \cap Q_2,$$

and hence

$$V_H(Q_1) \cup V_H(Q_2) = V_H(Q_1 \diamond Q_2) \supseteq V_H([Q_2, Q_2]) \supseteq V_H(Q_1 \cap Q_2).$$

The reverse inclusion is obvious.

Notice, that the $\diamond$-product plays exactly the same role as multiplication of polynomials in the case of polynomial algebras.

The upshot of the preceding lemma is

THEOREM 3. *Let H be a G-group. We define a subset of H^n to be closed if it is the intersection of an arbitrary number of finite unions of algebraic sets in*

H^n; this defines a topology on H^n, called the Zariski topology. The Zariski topology is a T_1-topology; i.e., the singletons are closed sets. Moreover if H is a G-domain, then the closed sets in the topology are the algebraic sets.

This kind of topology was first introduced by Bryant [BR] when $n = 1$.

Notice that continuity of maps in the Zariski topology depends only on the algebraic sets. The next lemma is an immediate consequence of this remark.

LEMMA 7. *Let H and K be G-groups and left $f: H^n \to K^m$ be a map. If the pre-image of an algebraic set of K^m is an algebraic set of H^n, then the map f is continuous in the Zariski topology.*

We remark that, in general, the union of two algebraic sets need not be algebraic.

EXAMPLE 4. *Let A be an abelian group, viewed as an A-group. Then any algebraic set in A^n is a coset with respect to some subgroup of A^n, where A^n is now an abelian group, under the operation of coordinatewise multiplication.*

Indeed, suppose $S \subseteq A[x_1, \dots, x_n]$. If $s \in S$, then s can be written in the form $s = s^*s'$, where s' is in the derived group of $A[x_1, \dots, x_n]$ and

$$s^* = x_1^{m(s)_1} x_2^{m(s)_2} \cdots x_n^{m(s)_n} a_s \qquad (a_s \in A). \tag{8}$$

Put $S^* = \{s^* | s \in S\}$. Since A is abelian $V_A(S) = V_A(S^*)$, which means that $V_A(S)$ consists exactly of all solutions in A^n of the multiplicatively written, linear system of equations

$$x_1^{m(s)_1} x_2^{m(s)_2} \cdots x_n^{m(s)_n} = a_s^{-1} \qquad (s \in S). \tag{9}$$

The algebraic set $V_A(S_1)$ of the corresponding homogeneous system S_1,

$$x_1^{m(s)_1} x_2^{m(s)_2} \cdots x_n^{m(s)_n} = 1 \qquad (s \in S), \tag{10}$$

is a subgroup of A^n and as usual

$$V_A(S) = bV_A(S_1),$$

where $b = (b_1, \dots, b_n) \in A^n$ is an arbitrarily chosen solution of the system of equations given by (9).

Since the union of two such cosets need not be a coset, for example the union of two distinct cosets with respect to the same subgroup, it follows that the union of two algebraic sets need not be an algebraic set.

3.2. *Ideals of Algebraic Sets*

Let, as before, H be a G-group, n be a positive integer, H^n be affine n-space over H and $G[X] = G[x_1, \dots, x_n]$. We recall some of the notions detailed in the Introduction.

DEFINITION 9. Let $Y \subseteq H^n$. Then

$$I_H(Y) = \{f \in G[X] | f(v) = 1 \text{ for all } v \in Y\}.$$

Notice, that $I_H(\emptyset) = G[X]$ and $1 \in I_H(Y)$ for every Y.

LEMMA 8. *For any $Y \subseteq H^n$ the set $I_H(Y)$ is an ideal of the G-group $G[x_1, \dots, x_n]$.*

Proof. In order to prove the lemma, observe that every point $v = (a_1, \dots, a_n) \in H^n$ can be used to define a G-homomorphism

$$\phi_v : G[X] \to H,$$

by *the evaluation map*,

$$\phi_v(f) = f(v), \qquad f \in G[X],$$

i.e., ϕ_v: $x_i \mapsto a_i$, $g \mapsto g$, where $i = 1, \dots, n$, $g \in G$. Now observe that

$$I_H(Y) = \bigcap_{v \in Y} \operatorname{Ker} \phi_v. \tag{11}$$

Since the intersection of ideals in a G-group is again an ideal, it follows that $I_H(Y)$ is an ideal of $G[X]$.

DEFINITION 10. An ideal in $G[X]$ is termed an H-closed ideal if it is of the form $I_H(Y)$ for some subset Y of H^n.

Sometimes we omit the subscript H and simply write $I(Y)$.

DEFINITION 11. If Y is an affine algebraic set in H^n, then $I(Y)$ is termed the ideal of Y.

The various parts of the following lemma are either consequences of the foregoing discussion or can be proved directly from the definitions.

LEMMA 9. *Let Y, Y_1, Y_2 be subsets of H^n. Then the following hold.*

1. $Y_1 \subseteq Y_2 \Rightarrow I(Y_1) \supseteq I(Y_2)$.
2. $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.
3. $I(Y_1 \cap Y_2) \supseteq I(Y_1) \cup I(Y_2)$.

4. $S \subseteq G[X] \Rightarrow I(V(S)) \supseteq \mathrm{gp}_{G[X]}(S)$.
5. *If Y is an algebraic set in H^n, then $V(I(Y)) = Y$.*
6. *If Q is an H-closed ideal of $G[X]$, then $I_H(V(Q)) = Q$.*

One consequence of Lemma 9 that is worth drawing attention to here is

COROLLARY 6. 1. *If Y_1 and Y_2 are algebraic sets in H^n, then*

$$Y_1 = Y_2 \quad \Leftrightarrow \quad I_H(Y_1) = I_H(Y_2).$$

2. *The functions I and V are inclusion reversing inverses when applied to the algebraic sets in H^n and the H-closed ideals in $G[X]$.*

The following lemma is of use in what follows.

LEMMA 10. *Suppose that the G-group H is a G-domain and that Y is a subset of H^n. Then $V(I(Y)) = \overline{Y}$, the closure of Y in the Zariski topology on H^n.*

Proof. Let C be a closed subset in H^n containing Y. The set C is an algebraic set for H is a G-domain. Then $I(C) \subseteq I(Y)$, and hence $C = V(I(C)) \supseteq V(I(Y))$. Therefore $V(I(Y))$ is the minimal closed subset of H^n containing Y, i.e., $V(I(Y)) = \overline{Y}$.

3.3. *Morphisms of Algebraic Sets*

Throughout this section we assume that H is a G-group, that $G[X] = G[x_1, \dots, x_n]$ and that H^n is affine n-space over H.
We make use of

DEFINITION 12. Let $f(x_1, \dots, x_n) \in G[X]$. The map $\mu_f\colon H^n \to H$ defined by

$$\mu(a_1, \dots, a_n) = f(a_1, \dots, a_n)$$

is termed the polynomial function on H^n defined by f. Its restriction to an algebraic set Y in H^n is similarly termed a polynomial function on Y.

LEMMA 11. *Any polynomial function $\mu_f\colon H^n \to H$ is continuous in the Zariski topology.*

Proof. By Lemma 7 it is enough to prove that the pre-image of any algebraic set $Z \subseteq H$ is an algebraic set in H^n. Now, if $Z = V_H(S)$ then

$$\mu_f^{-1}(Z) = V_H\big(s(f(x_1, \dots, x_n)) | s \in S\big).$$

Hence $\mu_f^{-1}(Z)$ is an algebraic set in H^n.

LEMMA 12. *Group multiplication and inversion in H are continuous in the Zariski topology.*

To prove this it suffices to notice that multiplication is described by $x_1 x_2$ and inversion by x_1^{-1}.

COROLLARY 7. *For any $a, b \in G$ the map $h \to ahb$ is a homeomorphism of H in the Zariski topology.*

We need next to define a morphism of algebraic sets.

DEFINITION 13. Let H be a G-group and let $Y \subseteq H^n$, $Z \subseteq H^p$ be algebraic sets. Then a map

$$\phi \colon Y \to Z$$

is termed a morphism of the algebraic set Y to the algebraic set Z if there exist $f_1, \dots, f_p \in G[x_1, \dots, x_n]$ such that for any $(a_1, \dots, a_n) \in Y$:

$$\phi(a_1, \dots, a_n) = \big(f_1(a_1, \dots, a_n), \dots, f_p(a_1, \dots, a_n)\big).$$

DEFINITION 14. Two algebraic sets Y and Z are said to be isomorphic if there exist morphisms

$$\phi \colon Y \to Z, \qquad \theta \colon Z \to Y,$$

such that $\theta\phi = 1_Y$, $\phi\theta = 1_Z$.

We then have

LEMMA 13. *Every morphism from an algebraic set $Y \subseteq H^n$ to an algebraic set $Z \subseteq H^p$ is a continuous map in the Zariski topology.*

The proof is similar to the proof of Lemma 11 and is therefore omitted.

COROLLARY 8. *The canonical projection $H^{n+m} \to H^m$ is continuous in the Zariski topology.*

3.4. *Coordinate Groups*

Let as before, H be a G-group, let $G[X] = G[x_1, \dots, x_n]$ and let $Y \subseteq H^n$ be an algebraic set defined over G.

We denote, as already stated in the Introduction, the set of all polynomial functions on Y by $\Gamma(Y)$. If $\mu, \nu \in \Gamma(Y)$, we define the product of μ and ν by

$$\mu\nu(g) = \mu(y)\nu(y) \qquad (y \in Y) \tag{12}$$

and the inverse of μ by

$$\mu^{-1}(y) = \mu(y)^{-1} \qquad (y \in Y). \tag{13}$$

PROPOSITION 9. *The set $\Gamma(Y)$ of all polynomial functions on Y forms a G-group with respect to multiplication and inversion, as defined above, with G embedded in $\Gamma(Y)$ via the mapping $g \to \mu_g$ of G into $\Gamma(Y)$ ($g \in G$).*

The proof is straightforward and is consequently omitted.

DEFINITION 15. $\Gamma(Y)$ is called the coordinate group of the algebraic set Y.

For each $i = 1, \dots, n$, define on $Y \subseteq H^n$ the polynomial function

$$t_i\colon (a_1, \dots, a_n) \mapsto a_i \ \big((a_1, \dots, a_n) \in Y\big).$$

These *coordinate functions* all belong to $\Gamma(Y)$. The following is an immediate consequence of the definitions.

LEMMA 14. *The map $x_1 \mapsto t_1, \dots, x_n \mapsto t_n$ defines a G-epimorphism from $G[X]$ onto $\Gamma(Y)$ with kernel $I_H(Y)$. Hence*

$$G[x_1, \dots, x_n]/I_H(Y) \simeq \Gamma(Y).$$

Denote

$$G_{R(Y)} = G[x_1, \dots, x_n]/I_H(Y).$$

In what follows we refer to the group $G_{R(Y)}$ also as the coordinate group of the algebraic set Y.

COROLLARY 9. *For any algebraic set $Y \subseteq H^n$ the coordinate group $\Gamma(Y)$ is G-separated by H.*

Proof. Since $\Gamma(Y)$ is G-isomorphic to $G[x_1, \dots, x_n]/I_H(Y)$ it suffices to prove that the latter group is G-separated by H. The separation now follows from the description of $I_H(Y)$ as the intersection of the kernels of G-homomorphisms from $G[x_1, \dots, x_n]$ into H (see Section 3.2).

In particular, if $G = H$, then the coordinate groups $\Gamma(Y)$ are all G-separated by G; i.e., they all are contained in the category $\mathcal{S}_G$ (see Section 2.4). We have more to say about this in Section 5.1.

3.5. *Equivalence of the Categories of Affine Algebraic Sets and Coordinate Groups*

Let H be a G-group and let $\mathbf{AS}_H$ be the category of all algebraic sets over H defined by systems of equations with coefficients in G (morphisms in $\mathbf{AS}_H$ are the morphisms of algebraic sets defined above). Denote by $\mathbf{AG}_H$ the category of all coordinate groups of the algebraic sets in $\mathbf{AS}_H$ (morphisms in $\mathbf{AG}_H$ are G-homomorphisms). Notice that both categories are defined relative to a given G-group H.

THEOREM 4. *Let H be a G-group. Then the categories $\mathbf{AS}_H$ and $\mathbf{AG}_H$ are equivalent to each other.*

We have to define two functors $\mathscr{F}$: $\mathbf{AS}_H \to \mathbf{AG}_H$ and $\mathscr{G}$: $\mathbf{AG}_H \to \mathbf{AS}_H$ such that

$$\mathscr{G}\mathscr{F} \simeq 1_{\mathbf{AS}_H}, \qquad \mathscr{F}\mathscr{G} \simeq 1_{\mathbf{AG}_H}.$$

If Y is an algebraic set defined over H, then $\mathscr{F}(Y)$ is the coordinate group of Y:

$$\mathscr{F}(Y) = \Gamma(Y).$$

In order to define $\mathscr{F}$ on morphisms, suppose that $Y \subseteq H^n$ and $Z \subseteq H^p$ are algebraic sets and ϕ: $Y \to Z$ is a morphism from Y to Z. We then define

$$\mathscr{F}(\phi): \Gamma(Z) \to \Gamma(Y),$$

as

$$\mathscr{F}(\phi)(f) = f \circ \phi,$$

where f is a polynomial function on Z (i.e., an element from $\Gamma(Z)$) and $\circ$ denotes the composition of functions. It is not hard to see that $\mathscr{F}(\phi)$ is a G-homomorphism of G-groups and that

$$\mathscr{F}(\psi\phi) = \mathscr{F}(\phi)\mathscr{F}(\psi), \qquad \mathscr{F}(1_Y) = 1_{\Gamma(Y)}.$$

Notice that $\mathscr{F}$ is a contravariant functor. We define next the functor $\mathscr{G}$: $\mathbf{AG}_H \to \mathbf{AS}_H$ as follows. Since the objects in $\mathbf{AG}_H$ are simply the $\Gamma(Y)$, where Y is an algebraic set in some H^n, we define

$$\mathscr{G}(\Gamma(Y)) = Y.$$

Next we define $\mathscr{G}$ on G-homomorphisms. To this end, let

$$\vartheta: \Gamma(Y) \to \Gamma(Z)$$

be a G-homomorphism from one coordinate group $\Gamma(Y)$ to another coordinate group $\Gamma(Z)$. Now we G-present each of $\Gamma(Y)$, $\Gamma(Z)$, using the isomorphisms

$$\Gamma(Y) \simeq G[x_1, \dots, x_n]/I(Y), \qquad \Gamma(Z) \simeq G[x_1, \dots, x_k]/I(Z).$$

So

$$\vartheta\big(x_i I(Y)\big) = w_i(x_1, \ldots, x_k) I(Z), \qquad i = 1, \ldots, n.$$

Define now $\mathscr{G}(\vartheta)\colon Z \to Y$ by

$$\mathscr{G}(\vartheta)\big((a_1, \ldots, a_k)\big) = \big(w_1(a_1, \ldots, a_k), \ldots, w_n(a_1, \ldots, a_k)\big).$$

It is not hard to verify then that $\mathscr{G}$ is a contravariant functor from $\mathbf{AG}_H$ to $\mathbf{AS}_H$. Moreover,

$$\mathscr{G}\mathscr{F} \simeq 1_{\mathbf{AS}_H}, \qquad \mathscr{F}\mathscr{G} \simeq 1_{\mathbf{AG}_H}.$$

COROLLARY 10. *Let H be a G-group. Then algebraic sets $Y_1 \subset H^n$ and $Y_2 \subset H^p$ are isomorphic if and only if the coordinate groups $\Gamma(Y_1)$ and $\Gamma(Y_2)$ are G-isomorphic.*

3.6. *The Zariski Topology of Equationally Noetherian Groups*

The equationally Noetherian property for a G-group H can be expressed in terms of the descending chain condition on algebraic sets over H.

PROPOSITION 10. *Let H be a G-group. Then H is G-equationally Noetherian if and only if every properly descending chain of algebraic sets over H is finite.*

Proof. Suppose that H is G-equationally Noetherian. Every strictly descending chain of algebraic sets in H^n

$$V_1 \supset V_2 \supset \cdots \tag{14}$$

gives rise to a strictly ascending chain of ideals:

$$I(V_1) \subset I(V_2) \subset \cdots. \tag{15}$$

Put

$$S = \bigcup_i I(V_i).$$

Then $v_H(S) = V_H(S_0)$ for some finite subset S_0 of S. But then $S_0 \subseteq I(V_m)$ for some m, which implies that

$$\bigcap_i V_i = V_H(S) = V_H\big(I(V_m)\big) = V_m;$$

i.e., the chain (14) terminates in no more than m steps.

Assume now that the set of all algebraic sets over H satisfies the descending chain condition. Let S be a subset of $G[x_1, \ldots, x_n]$. If $V_H(S) \neq V_H(S_0)$ for any finite subset S_0 of S, then there exists an infinite sequence $s_1, s_2, \ldots$ of elements of S such that

$$V_H(s_1) \supset V_H(s_1, s_2) \supset \cdots$$

is an infinite, strictly descending chain of algebraic sets, a contradiction. Hence H is G equationally Noetherian.

We recall that a topological space is termed *Noetherian* if it satisfies the descending chain condition on closed subsets.

THEOREM D1. *Let H be a G-group. Then for each integer $n > 0$, the Zariski topology on H^n is Noetherian if and only if H is G-equationally Noetherian.*

Proof. Suppose that the G-group H is G-equationally Noetherian. We need to prove that H^n is Noetherian for every $n > 0$ in the Zariski topology. It follows from Proposition 10 that the set A of all algebraic sets contained in H^n satisfies the descending chain condition. Let A_1 be the set of all finite unions of the sets in A and let A_2 be the set of all (possibly infinite) intersections of sets in A_1. By the definition of the Zariski topology, A_2 is the set of closed subsets of H^n. We first prove that A_1 satisfies the descending chain condition. Suppose that $M_1 = V_1 \cup \cdots \cup V_m$ and that $M_2 = W_1 \cup \cdots \cup W_k$ are sets in A_1 and that $M_1 \supset M_2$. Then for every $i \leq m$ we have $V_i \supset V_i \cap W_j$ which gives rise to a tree of subsets with root vertex V_i and with a unique edge from the root to every proper subset of the form $V_i \cap W_j$. A strictly descending chain of sets in A_1, say

$$M_1 \supset M_2 \supset \cdots, \tag{16}$$

gives rise to m trees of subsets such that each vertex of each tree is a finite intersection of sets in A, hence in A; moreover, for each such vertex there are only finite many outgoing edges. In the resultant graph, every path corresponds to a strictly descending chain of algebraic sets and so is finite. By Koenig's lemma this implies that the whole graph is finite. Therefore, the chain (16) is also finite.

Since A_1 satisfies the descending chain condition and is closed under finite intersections, the intersection of an arbitrary collection of sets in A_1 is the intersection of some finite subcollection; hence it is also in A_1. Consequently, $A_2 = A_1$ and hence satisfies the descending chain condition. Consequently, H^n is Noetherian in the Zariski topology. Conversely, if H^n is Noetherian for every n, then H is G-equationally Noetherian by Proposition 10. This completes the proof.

COROLLARY 11. *Let H be a G-equationally Noetherian group. Then every closed set in* H^n *is a finite union of algebraic sets.*

A non-empty subset Y of a topological space X is said to be *irreducible* if it cannot be expressed as the union $Y = Y_1 \cup Y_2$ of two proper subsets, each one of which is closed in Y, in the induced topology.

PROPOSITION 11. *In a Noetherian topological space X every non-empty closed subset Y can be expressed as a finite union* $Y = Y_1 \cup \cdots \cup Y_n$ *of irreducible closed subsets* Y_i. *If we require that* $Y_i \not\subseteq Y_j$ *for* $i \neq j$, *then the* Y_i, *the so-called irreducible components of Y, are unique.*

The proof is standard (see, for example, [HR]).

COROLLARY 12. *Let H be a G-equationally Noetherian group. Then every subset Y of* H^n, *which is closed in the Zariski topology, is a finite union of irreducible algebraic sets, each of which is uniquely determined by Y.*

PROPOSITION 12. *Let H be a G-equationally Noetherian group and let Y be a subset of* H^n, *closed in the Zariski topology. If* $Y = Y_1 \cup \cdots \cup Y_m$ *is the decomposition of Y into its irreducible components, then the coordinate group* $\Gamma(Y)$ *is canonically embedded into the direct product of the coordinate groups* $\Gamma(Y_i)$:

$$\Gamma(Y) \hookrightarrow \Gamma(Y_1) \times \cdots \times \Gamma(Y_m).$$

Proof. The irreducible decomposition $Y = Y_1 \cup \cdots \cup Y_m$ implies (by Lemma 9) the corresponding decomposition of the ideal $I(Y)$:

$$I(Y) = I(Y_1) \cap \cdots \cap I(Y_m).$$

The canonical homomorphisms

$$\lambda_i: \Gamma(Y) = G[X]/I(Y) \to G[X]/I(Y_i) \simeq \Gamma(Y_i)$$

give rise to an embedding

$$\lambda: \Gamma(Y) \to \Gamma(Y_1) \times \cdots \times \Gamma(Y_m),$$

where $\lambda = \lambda_1 \times \cdots \times \lambda_m$.

The following lemma establishes a very important property of coordinate groups of irreducible closed sets.

LEMMA 15. *Let H be a G-equationally Noetherian group. If a closed set* $Y \subseteq H^n$ *is irreducible, then the coordinate group* $\Gamma(Y)$ *is G-discriminated by H.*

Proof. Let $S \subseteq G[X]$ and $Y = V_H(S)$. As we mentioned above $\Gamma(Y) \simeq G[X]/I(S)$. Suppose that there exist finitely many non-trivial elements $u_1 I(S), \dots, u_n I(S)$ in $G[X]/I(S)$ that cannot be discriminated in H by a G-homomorphism; i.e., for any G-homomorphism ϕ: $G[X]/I(S) \to H$ there exist an i such that $\phi(u_i I(S)) = 1$. It follows then that

$$Y = V_H(S) = V_H(S \cup \{u_1\}) \cup \cdots \cup V_H(S \cup \{u_n\}).$$

On the other hand, since $G[X]/I(S)$ is G-separated by H (see Section 3.4 Corollary 8), $V_H(S \cup \{u_i\})$ is a proper closed subset of $V_H(S) = Y$. This contradicts the irreducibility of Y.

In particular, if $G = H$ is equationally Noetherian, then the coordinate group $\Gamma(Y)$ of an irreducible closed set $Y \subseteq G^n$ is G-discriminated by G; i.e., $\Gamma(Y)$ belongs to the category $\mathscr{D}_G$ from Section 2.4.

In the case when H is a G-equationally Noetherian G-domain we have the following important characterization of irreducible closed sets in terms of their coordinate groups.

THEOREM 5. *Let H be a G-equationally Noetherian G-domain. Then a closed subset $Y \subseteq H^n$ is irreducible if and only if the coordinate group* $\Gamma(Y)$ *is G-discriminated by H.*

Proof. The only if statement has already been proved in Lemma 15 above. Now, let the coordinate group $\Gamma(Y)$ of a closed set $Y \subseteq H^n$ be G-discriminated by H. Suppose

$$Y = Y_1 \cup \cdots \cup Y_m$$

is the decomposition of Y into its irreducible components, then

$$I(Y) = I(Y_1) \cap \cdots \cap I(Y_m),$$

and by Proposition 12 there exists an embedding

$$\lambda\colon \Gamma(Y) \to \Gamma(Y_1) \times \cdots \times \Gamma(Y_m),$$

which is induced by the canonical epimorphisms

$$\lambda_i\colon \Gamma(Y) = G[X]/I(Y) \to G[X]/I(Y_i) \simeq \Gamma(Y_i).$$

We claim that at least one of these canonical epimorphisms is an isomorphism. Indeed, suppose that each epimorphism λ_i has a non-trivial kernel. Choose an arbitrary non-trivial element u_i from the kernel of λ_i, $i = 1, \dots, m$. The group $\Gamma(Y)$ is G-discriminated by the G-domain H; hence $\Gamma(Y)$ is also a G-domain (see Section 2.3). Therefore, there are elements

$g_2, \ldots, g_m \in G$ such that the commutator

$$u = [u_1, u_2^{g_2}, \ldots, u_m^{g_m}]$$

is non-trivial. Hence $\lambda(u) \neq 1$. But, for each $i = 1, \ldots, m$, $\lambda_i(u) = 1$, which implies that $\lambda(u) = 1$, contradicting the observation above. Hence, for a suitable choice of i, the homomorphism λ_i is an isomorphism. Consequently, $I(Y) = I(Y_i)$ and hence $Y = Y_i$, which implies that Y is irreducible.

4. IDEALS

4.1. *Maximal Ideals*

Throughout this section let G denote a *non-trivial* group and let H be a G-group.

Ideals in G-groups enjoy many of the same properties as do ideals in commutative rings. We term an ideal M of the G-group H *a maximal ideal* of H if M is not contained in any other ideal of H. Notice that in view of the fact that $G \neq 1$, every ideal of H is different from H.

LEMMA 16. *If $G \neq 1$ then every ideal in a G-group H is contained in a maximal ideal.*

The proof of the Lemma is a standard application of Zorn's lemma.

DEFINITION 16. A G-group H is termed G-simple, if the only proper ideal of H is the ideal $\{1\}$.

Notice, that any group G is G-simple. New examples of G-simple groups come surprisingly from nilpotent groups. Indeed, let H be a non-trivial nilpotent group and G be the center of H; then H is G-simple. Observe, that an ideal M of the G-group H is maximal if and only if H/M is G-simple.

We focus first on maximal ideals in $G[X]$. The following is an easy but important lemma.

LEMMA 17. *If $a = (a_1, \ldots, a_n) \in G^n$, then the ideal $I_G(a)$ is a maximal ideal of $G[X]$, corresponding to the point $a \in G^n$.*

Proof. Observe that

$$I_G(a) = \mathrm{gp}_{G[X]}\{x_1 a_1^{-1}, \ldots, x_n a_n^{-1}\},$$

and so the factor group

$$G[X]/I_G(a) \simeq G$$

is G-simple; hence $I_G(a)$ is maximal.

In general, if H is a G-group and $a \in H^n$, then $I_H(a)$ is not necessarily a maximal ideal of $G[X]$ since every ideal of $G[X]$ can be represented in this way—all we have to do is to choose the G-group H suitably.

LEMMA 18. *Let Q be an ideal in $G[X]$. Then the following hold*:

1. $Q = I_H(a)$, *where* $a = (x_1Q, \dots, x_nQ) \in H^n$ *with* $H = G[X]/Q$;
2. *If M is a maximal ideal in $G[X]$ and a is a root of M in some G-group H, then a is the only root of M in H and $M = I_H(a)$.*

Proof. The first statement follows from the definition of the quotient $G[X]/Q$. To prove the second one, it suffices to notice that if $a \in V_H(M)$ then $M \subseteq I_H(a)$ and that $V_H(I_H(a)) = \{a\}$.

So, in particular, every maximal ideal M in $G[X]$ is of the form $M = I_H(a)$ for some tuple $a \in H^n$, where H is a supergroup of G. However a need not lie in G^n. This leads to the idea of replacing G by some "universal completion," say $\overline{G}$, of G which contains a root of every ideal in $G[X]$. In classical algebraic geometry such completions are, of course, just algebraically closed fields. In Section 2.4 we have introduced and studied $\aleph_0$-universal groups in the categories $\mathcal{S}_G$ and $\mathcal{D}_G$ which play a role similar to that of algebraically closed fields. For the moment it suffices to mention only the following consequence of Theorem G1, which we have mentioned in the Introduction.

COROLLARY 13. *Let G be an algebraically closed group. Then an ideal in $G[X]$, which is the normal closure of a finite set, is maximal if and only if it is of the form $I_G(a)$, where $a \in G^n$.*

As we remarked before, not all ideals of $G[X]$ of the type $I_H(a)$, $a \in H^n$, are maximal, since by Lemma 18 all ideals are of this form. Similarly, not all maximal ideals in $G[X]$ are of the form $I_G(a)$, $a \in G^n$. The following examples illustrate some of the possibilities that can occur.

EXAMPLE 5. *Let $H = \langle x_1, \dots, x_n; R \rangle$ be a presentation of a finitely generated simple group and let g be an element of H of infinite order. Let G be the subgroup generated by g. Then H is a G-group. Let ϕ be the obvious G-homomorphism of $G[X]$ onto H. Then the kernel M of ϕ is an ideal of $G[X]$ with $G[X]/M \simeq H$. Since H is simple and hence G-simple, it follows that M is a maximal ideal in $G[X]$. But $G[X]/M$ is not isomorphic to G in the category of G-groups, which means that M is not of the form $I_G(a)$ with $a \in G^n$.*

Another example of this kind is the following one.

EXAMPLE 6. *Let F be a free group freely generated by c and d and let $Q = \mathrm{gp}_{F[x]}([c,d]x^2)$. Then Q is an ideal of $F[x]$, but no maximal ideal in $F[x]$ containing Q has any F-points and so it is never of the form $I_F(a)$, $a \in F^1$.*

Proof. Put $H = F[x]/Q$. Then H is a free product with amalgamation of F and the infinite cyclic group generated by x: $H = F *_{[c,d]=x^2} \langle x \rangle$. Hence H is an F-group and therefore Q is an ideal in $F[x]$. Consequently, Q is contained in some maximal ideal M of $F[x]$. Since the equation $[c,d] = x^2$ has no solutions in F, M has no F-points.

We have seen that there are two different types of maximal ideals in $G[X]$: ones that have G-points and those that do not. Following on the procedure in ring theory we call ideals of the first type *G-rational ideals* or *G-maximal ideals*. They play an important role in the next section.

The following corollary is a consequence of the remarks above.

COROLLARY 14. *Let M be a maximal ideal in $G[X]$, where $G \neq 1$. Then the following conditions are equivalent*:

1. *M is G-maximal*;
2. *$M = I_G(a)$ for some $a \in G^n$*;
3. *M has a G-point*;
4. *$G[X]/M \simeq G$.*

4.2. *Radicals*

In this section we introduce the notion of a radical ideal of an arbitrary G-group which is the counterpart of the notion of a closed ideal in a free G-group $G[X]$.

Let H be a G-group with $G \neq 1$.

DEFINITION 17. The Jacobson G-radical $J_G(H)$ of a G-group H is the intersection of all G-maximal ideals in H; if there are no such ideals, we define $J_G(H) = H$.

Similarly, we define the G-radical $\mathrm{Rad}_G(Q)$ of an arbitrary ideal Q in H as follows.

DEFINITION 18. Let Q be an ideal in a G-group H. Then the G-radical $\mathrm{Rad}_G(Q)$ of Q is the intersection of all G-maximal ideals in H containing Q; if there are no such ideals, we define $\mathrm{Rad}_G(Q) = H$.

We term Q a G-radical ideal of H if $\mathrm{Rad}_G(Q) = Q$.

LEMMA 19. *Let H be a G-group and let Q be an ideal in H. Then for an ideal P of H the following conditions are equivalent*:

1. $P = \mathrm{Rad}_G(Q)$;
2. *P is the pre-image in H of $J_G(H/Q)$*;
3. *P is the smallest ideal in H such that P contains Q and the quotient group H/P is G-separated by G.*

The proof is easy and we leave it to the reader.

PROPOSITION 13. *Let G be a non-abelian torsion-free hyperbolic group. Then*

$$J_G(G[X]) = 1.$$

Proof. We proved in [BMR2] that the free G-group $G[X]$ is G-discriminated by G provided G is non-abelian and torsion-free hyperbolic.

Groups H with $J_G(H) = 1$ are very important because of their close relationship to coordinate groups of algebraic sets over G. We say more about this in Section 5.1.

We need the following generalization of the notion of G-radical.

DEFINITION 19. Let H and K be G-groups. Then the Jacobson K-radical $J_K(H)$ is the intersection of the kernels of all G-homomorphisms from H into K.

If Q is an ideal of H, then the K-radical $\mathrm{Rad}_K(Q)$ of Q in H is the pre-image in H of $J_K(H/Q)$.

We term an ideal Q of H K-radical if $Q = \mathrm{Rad}_K(Q)$.

Clearly, $\mathrm{Rad}_K(Q)$ is the smallest ideal P in H containing Q and such that H/P is G-separated by K.

LEMMA 20. *The intersection of two K-radical ideals in a G-group H is a K-radical ideal.*

Proof. Let Q_1 and Q_2 are two arbitrary K-radical ideals in H. Clearly, every non-trivial element of the group $H/(Q_1 \cap Q_2)$ can be separated by the canonical epimorphism either onto the group H/Q_1 or onto the group H/Q_2. Both of these groups are G-separated in K, therefore $H/(Q_1 \cap Q_2)$ is G-separated in K. Hence the ideal $Q_1 \cap Q_2$ is K-radical.

In the following lemma we describe radical ideals of the free G-group $G[X]$. We recall that an ideal Q of $G[X]$ is K-closed if $Q = I_K(Y)$ for some subset Y of K^n (here K is an arbitrary G-group).

PROPOSITION 14. *Let Q be an ideal in $G[X]$. Then for an arbitrary G-group K the following statements are equivalent*:

1. *Q is K-closed in $G[X]$;*
2. $Q = \mathrm{Rad}_K(Q)$;
3. *G/Q is G-separated by K.*

Proof. As we have already mentioned above, the equivalence of 2. and 3. is a direct consequence of the definitions.

Now we prove that 1. implies 3. Let Q be a K-closed ideal in $G[X]$. Then $Q = I_K(Y)$ for some subset $Y \subseteq K^n$. Hence $G[X]/Q = \Gamma(Y)$ and the result follows from Corollary 9, Section 3.4.

To finish the proof it suffices to show that 3. implies 1. Suppose that $G[X]/Q$ is G-separated by K. We claim that $I_K(V_K(Q)) = Q$, which shows that Q is K-closed. To this end, if ϕ is a G-homomorphism from $G[X]/Q$ into K, let $\phi(x_i Q) = a_i$ for each $i = 1, \dots, n$. If $h(x_1, \dots, x_n) \in Q$, then

$$1 = \phi(Q) = \phi(hQ) = h(a_1, \dots, a_n).$$

It follows that $V_K(Q) \neq \emptyset$. Now if $f \in G[X]$ and if $f \notin Q$, then we can find a homomorphism θ_f: $G[X]/Q \to K$ such that $\theta_f(fQ) \neq 1$. So if we put $b_i = \theta_f(x_i Q)$, then $b = (b_1, \dots, b_n) \in V_K(Q)$ but $f(b_1, \dots, b_n) = \theta_f(fQ) \neq 1$. It follows that we have proved that $I_K(V_K(Q)) \subseteq Q$ and therefore that $I_K(V_K(Q)) = Q$, as needed.

The next lemma provides another relationship between K-radical ideals of G-groups and K-closed ideals in $G[X]$.

LEMMA 21. *Let H be a G-group and η: $G[X] \to H$ be a G-epimorphism from a the free G-group $G[X]$ onto H. Then for any ideal Q in H and any G-group K the following conditions are equivalent*:

1. *Q is K-radical in H;*
2. *$\eta^{-1}(Q)$ is K-closed in $G[X]$.*

Proof. It follows from the appropriate isomorphism theorems that $\eta^{-1}(Q)$ is an ideal in $G[X]$ and

$$H/Q \simeq G[X]/\eta^{-1}(Q).$$

Therefore G-separability of one of the groups above implies G-separability of the other. The result then follows from Proposition 14.

The next proposition details the connection between the equationally Noetherian condition and the ascending chain condition on radical ideals.

PROPOSITION 15. *Let H be a finitely generated G-group. Then H satisfies the ascending chain condition on K-radical ideals for each G-equationally Noetherian group K.*

Proof. Let

$$Q_1 \subset Q_2 \subset \cdots \tag{17}$$

be a properly ascending chain of G-radical ideals in H. Let $\eta: G[X] \to H$ be an epimorphism from a suitably chosen finitely generated free G-group $G[X]$ onto H. Observe, that if Q is a K-radical ideal in H then $\eta^{-1}(Q)$ is a K-radical ideal in $G[X]$. Hence

$$\eta^{-1}(Q_1) \subset \eta^{-1}(Q_2) \subset \cdots \tag{18}$$

is a properly ascending chain of K-radical ideals in $G[X]$. Therefore the algebraic sets defined by these ideals give rise to a properly descending chain of algebraic sets in K^n:

$$V_G\big(\eta^{-1}(Q_1)\big) \supset V_G\big(\eta^{-1}(Q_2)\big) \supset \cdots. \tag{19}$$

Since K is G-equationally Noetherian, this chain terminates, and therefore so do (18) and (17).

Notice that it follows directly from the definitions, that if H and K are G-groups and Q is an ideal of H, then the following inclusions hold

$$\mathrm{Rad}_G(Q) \supseteq \mathrm{Rad}_K(Q) \supseteq \mathrm{Rad}_{H/Q}(Q) = Q.$$

But if the group K is G-separated by G (i.e., if $K \in \mathscr{S}_G$) then $\mathrm{Rad}_K(Q) = \mathrm{Rad}_G(Q)$.

We formulate next the following definition.

DEFINITION 20. Let H be a G-group and let Q be an ideal of $G[X]$. Then a point $h = (h_1, \dots, h_n) \in H^n$ is termed a generic point of $V_H(Q)$ if

$$f \in I\big(V_H(Q)\big) \quad \Leftrightarrow \quad f(h) = 1.$$

We have already seen that the G-group $G[X]/Q$ plays a special role here. The following lemma is an amplification of the definition above and this remark. It follows immediately from the definitions.

LEMMA 22. *Let Q be an ideal of $G[X]$. For any G-group H containing the group $G[X]/Q$ the point $h = (x_1Q, \dots, x_nQ) \in H^n$ is a generic point of $V_H(Q)$.*

Proof. Notice, that $h = (x_1Q, \dots, x_nQ) \in V_H(Q)$. Now let $f \in G[x_1, \dots, x_n]$. Then $f(x_1Q, \dots, x_nQ) = 1$ in H if and only if $f(x_1Q, \dots, x_nQ)$

$= 1$ in $G[X]/Q$ which is equivalent to $f(x_1, \ldots, x_n) \in Q$. Therefore

$$f \in I(V_H(Q)) \quad \Leftrightarrow \quad f(h) = 1,$$

as desired.

Our next step is to define the *G-nilradical* of a G-group.

DEFINITION 2.1. Let H be a G-group. The subgroup $L_G(H)$ of H generated by all locally nilpotent ideals in H is termed the G-nilradical of H.

In the case where $G = 1$, $L_1(H)$ is usually referred to as the Hirsch–Plotkin radical of H.

Clearly every element in $L_G(H)$ is a G-nilpotent element. Hence, if H has no G-zero divisors then $L_G(H) = 1$.

PROPOSITION 16. *Suppose that the Hirsch–Plotkin radical $L_1(G)$ of G is trivial. Then for any G-group H, the G-nilradical $L_G(H)$ is an ideal in H which is contained in every maximal ideal of H and therefore*

$$J_G(H) \supseteq L_G(H).$$

Proof. If $L_1(G) = 1$, then the intersection $G \cap L_G(H)$ is normal locally nilpotent subgroup of G, hence it is trivial. Consequently $L_G(H)$ is an ideal. Suppose now that $L_G(H) \nsubseteq M$, for some maximal ideal M in the G-group H. Then there exists an element $c \in L_G(H)$ such that $c \notin M$. Then $\mathrm{gp}_H(c)$ is a locally nilpotent, normal subgroup of H. Now $M\,\mathrm{gp}_H(c)/M$ is a locally nilpotent, normal subgroup of the G-group H/M. Hence it meets GM/M (which is isomorphic to G) trivially. Therefore $M\,\mathrm{gp}_H(c)$ meets G trivially and so is an ideal of H which properly contains M, contradicting the maximality of M. This proves the proposition.

Finally we have the

PROPOSITION 17. *Let H be a G-equationally Noetherian G-group. Then $L_G(H)$ is a solvable subgroup of H.*

Proof. An equationally Noetherian group is a CZ-group (see [WB] for definitions). Therefore $L_G(H)$ is a locally nilpotent CZ-group. Hence it is solvable [WB].

4.3. *Irreducible and Prime Ideals*

We recall here some of the definitions from the Introduction.

DEFINITION 22. 1. We term the ideal Q of H irreducible if $Q = Q_1 \cap Q_2$ implies that either $Q = Q_1$ or $Q = Q_2$, for any ideals Q_1 and Q_2 of H.

2. An ideal Q of H is a prime ideal if H/Q is a G-domain.

The following proposition ties the irreducibility of algebraic sets with the irreducibility of their ideals.

PROPOSITION 18. *Let H be a G-group. Then the following hold*:

1. *If Q is an irreducible H-closed ideal in $G[X]$, then $V_H(Q)$ is an irreducible closed set in H^n*;

2. *If H is G-domain and Y is an irreducible closed set in H^n, then $I_H(Y)$ is an irreducible H-closed ideal in $G[X]$.*

Proof. 1. Suppose that $V_H(Q) = Y_1 \cup Y_2$, where Y_1 and Y_2 are closed sets in H^n. Then

$$Q = I_H\big(V_H(Q)\big) = I_H(Y_1 \cup Y_2) = I_H(Y_1) \cap I_H(Y_2).$$

Consequently, by the irreducibility of Q, either $Q = I_H(Y_1)$ or $Q = I_H(Y_2)$. It follows that either

$$V_H(Q) = V_H\big(I_H(Y_1)\big) = Y_1 \quad \text{or else} \quad V_H(Q) = V_H\big(I_H(Y_2)\big) = Y_2.$$

2. Suppose that $I_H(Y) = Q_1 \cap Q_2$ is the intersection of two ideals Q_1 and Q_2. Then by Lemma 6,

$$Y = V_H\big(I_H(Y)\big) = V_H(Q_1 \cap Q_2) = V_H(Q_1) \cup V_H(Q_2),$$

(the last equality holds since H is a G-domain). Hence, by the irreducibility of Y, either $Y = V_H(Q_1)$ or else $Y = V_H(Q_2)$. It suffices to consider the first possibility. Then $I_H(Y) = I_H(V_H(Q_1)) \supseteq Q_1$ and therefore $Q_1 = I_H(Y)$. It follows that $I_H(Y)$ is irreducible. This completes the proof.

PROPOSITION 19. *Let H be a G-group. Then the following hold*:

1. *If Q is a prime ideal of H, then Q is irreducible.*

2. *If Q is an irreducible ideal of H which is K-radical for some G-domain K, then Q is a prime ideal.*

Proof. 1. Suppose that the prime ideal Q is not irreducible. Then we can write $Q = Q_1 \cap Q_2$, where neither Q_1 nor Q_2 is contained in Q. Choose $c_1 \in Q_1 - Q$ and $c_2 \in Q_2 - Q$. Then modulo Q, $[c_1, c_2^g] \equiv 1$ for every $g \in G$. But then $c_1 Q$ is a G-zero divisor in H/Q, which is impossible.

2. Since Q is K-closed in H,

$$Q = \bigcap_{\phi \in \mathrm{Hom}_G(H/Q, K)} \ker \phi,$$

where $\mathrm{Hom}_G(H/Q, K)$ is the set of all G-homomorphisms from H/Q into K. Suppose that Q is not prime. Then there exist $c_1, c_2 \in H - Q$ such that $[c_1, c_2^g] \in Q$, for every $g \in G$. Then $[c_1, c_2^g] \in \ker \phi$ for every $\phi \in \mathrm{Hom}_G(H/Q, K)$. But $H/\ker \phi$ is G-isomorphic to a G-subgroup of the G-domain K and so is itself a G-domain. Therefore, either $c_1 \in \mathrm{Ker}\, \phi$ or else $c_1 \in \mathrm{Ker}\, \phi$. Let Φ_i be the set of all those $\phi \in \mathrm{Hom}_G(H/Q, K)$ for which $c_i \in \ker \phi$, $i = 1, 2$. Put

$$Q_i = \bigcap_{\phi \in \Phi_i} \ker \phi \qquad (i = 1, 2).$$

Then Q_1 and Q_2 are ideals in H and $Q = Q_1 \cap Q_2$. Since Q is irreducible, we find that either $Q = Q_1$ or $Q = Q_2$, which implies that either $c_1 \in Q$ or $c_2 \in Q$. It follows that Q is a prime ideal in H.

COROLLARY 15. *If H is a G-domain, then every irreducible H-closed ideal Q of $G[X]$ is prime.*

Indeed, as we mentioned in the previous section every H-closed ideal of $G[X]$ is an H-radical ideal of $G[X]$. The result now follows from the lemma above.

The next proposition is useful.

PROPOSITION 20. *Let H be a G-group. If Q is an ideal of H, then Q is prime and K-radical for some G-domain K if and only if H/Q is G-discriminated by K.*

Proof. By definition an ideal Q in H is K-radical if and only if G/Q is separated by K. In this event if K is a G-domain then by Theorem C1 H/Q is G-discriminated by K.

Now suppose that H/Q is G-discriminated by K and K is a G-domain. If Q is not prime in H then H/Q has a pair of non-trivial elements c_1 and c_2 such that $[c_1, c_2^g] = 1$ for every $g \in G$. Since H/Q is G-discriminated by K there exists a G-homomorphism $\phi\colon H/Q \to K$ such that $\phi(c_1) \neq 1$ and $\phi(c_2) \neq 1$. Since $[\phi(c_1), \phi(c_2)^g] = 1$ for every $g \in G$, it follows that $\phi(c_1)$ is a G-zero divisor in K, which is impossible. This completes the proof of the proposition.

4.4. *Decomposition Theorems*

We are now in a position to prove Theorem E1.

THEOREM E1. *Let H be a G-equationally Noetherian G-domain. Then each H-closed ideal in $G[X]$ is the intersection of finitely many prime H-closed ideals, none of which is contained in any one of the others and this representation is unique up to order. Consequently, distinct irredundant intersections of prime H-closed ideals define distinct H-closed ideals.*

Proof. Let Q be an H-closed ideal in $G[X]$. Let $Y = V_H(Q)$. Since H is a G-equationally Noetherian G-domain, Y can be decomposed uniquely into a finite union of irreducible algebraic sets:

$$Y = Y_1 \cup \cdots \cup Y_m.$$

This gives rise to the decomposition

$$Q = I(Y) = I(Y_1) \cap \cdots \cap I(Y_m).$$

By Proposition 18, each of the $I(Y_j)$ is irreducible. Now by Corollary 15 each of the ideals $I(Y_j)$ is prime. That proves the existence of the irredundant decomposition of Q. Moreover, if $Q = Q_1 \cap \cdots \cap Q_k$ is a decomposition of Q as an intersection of irreducible H-closed ideals, none of which is contained in any one of the other, then

$$Y = V_H(Q) = V_H(Q_1) \cup \cdots \cup V_H(Q_k)$$

is a irredundant decomposition of Y into irreducible closed sets. It follows that $k = m$ and that the $V_H(Q_j)$ are simply a rearrangement of the Y_l and hence that the Q_j are simply the $I(Y_l)$, in a possibly different order. To finish the proof it suffices to notice that the intersection of two (or finitely many) H-radical ideals is again H-radical—this was proved in Section 4.2, Lemma 20.

Theorem E1 has a counterpart for the ideals of finitely generated G-groups.

THEOREM E2. *Let H be a finitely generated G-group and let K be a G-equationally Noetherian G-domain. Then each K-radical ideal in H is a finite irredundant intersection of prime K-radical ideals. Moreover, this representation is unique up to order. Furthermore, distinct irredundant intersections of prime K-radical ideals define distinct K-radical ideals.*

Proof. Let Q be a K-radical ideal in H. Then $J_K(H/Q) = 1$. The group H/Q is finitely generated as a G-group, so we can express H/Q as a factor group of a finitely generated free G-group $G[X]$, say,

$$H/Q = G[X]/P.$$

Since $J_K(G[X]/P) = J_K(H/Q) = 1$, the ideal P is K-radical. By Proposition 14 from Section 4.2 the ideal P is K-closed in $G[X]$. Hence, by Theorem E1, P can be expressed as a finite intersection of prime K-closed ideals in $G[X]$:

$$P = P_1 \cap \cdots P_m.$$

We claim that each such prime decomposition of P in $G[X]$ gives rise to a prime decomposition of Q in H. Indeed, let Q_i be the pre-image of the ideal P_i/P in H with respect to the canonical G-epimorphism

$$H \to H/Q \simeq G[X]/P.$$

Then Q_i is K-radical and prime in H. Observe, that

$$Q = Q_1 \cap \cdots \cap Q_m.$$

Similarly, each prime decomposition of Q in H gives rise to a prime decomposition of P in $G[X]$. The uniqueness of the decomposition follows from Theorem E1.

In the case $K = G$ we have

COROLLARY 16. *Let G be an equationally Noetherian domain and let H be a finitely generated G-group. Then each G-radical ideal in H is a finite irredundant intersection of prime G-radical ideals. Moreover, this representation is unique up to order. Furthermore, distinct irredundant intersections of prime G-radical ideals define distinct G-radical ideals.*

5. COORDINATE GROUPS

5.1. *Abstract Characterization of Coordinate Groups*

In this section we describe coordinate groups in purely group-theoretic terms.

PROPOSITION 21. *Let H be a finitely generated G-group. If $J_K(H) = 1$ for some G-group K, then H is G-isomorphic to the coordinate group $\Gamma(Y)$ of some algebraic set $Y \subseteq K^n$ defined over G. Conversely, every coordinate group $\Gamma(Y)$ of an algebraic set $Y \subseteq K^n$ defined over G, is a finitely generated G-group with $J_K(\Gamma(Y)) = 1$.*

Proof. Suppose that H is a finitely generated G-group with $J_K(H) = 1$, where K is a given G-group. We express H as factor group of a finitely G-generated G-free group $G[X]$:

$$H \simeq G[X]/Q.$$

Thus $J_K(G[X]/Q) = 1$, i.e., $\mathrm{Rad}_K(Q) = Q$. Consequently by Proposition 14 from Section 4.2 Q is K-closed in $G[X]$ and so H is isomorphic to the coordinate group of the algebraic set $V_K(Q)$ defined over G.

Let $\Gamma(Y)$ be the coordinate group of some algebra set $Y \subseteq K^n$. So $\Gamma(Y) \cong H = G[X]/Q$, where $Q = I_K(Y)$. Again by Proposition 14 from

Section 4.2, $Q = \mathrm{Rad}_K(Q)$, hence

$$J_K(\Gamma(Y)) = J_K(G[X]/Q) = 1,$$

as desired.

Proposition 21 demonstrates again the role of the category $\mathscr{S}_G$ in algebraic geometry over groups, in particular, the importance of $\aleph_0$-universal groups from $\mathscr{S}_G$.

COROLLARY 17. *Let H be a G-group. Then the coordinate groups of algebraic sets in H^n defined over G are exactly the n-generator G-groups which are G-separated by H.*

In the event that $H = G$ this corollary shows that the coordinate groups of the algebraic sets over G are exactly the finitely generated G-groups in the category $\mathscr{S}_G$; in particular, they are exactly the finitely generated G-subgroups of the unrestricted direct power $G_{\aleph_0}(I)$ of I copies of G (see Section 2.4 for details) provided that the set of indices I is sufficiently large.

In [BMR2] we gave a large number of examples of finitely generated groups in $\mathscr{S}_G$; here it is worthwhile to record one such example.

COROLLARY 18. *If G is non-abelian torsion-free hyperbolic group, then $G[x_1, \ldots, x_n]$ is the coordinate group of the algebraic set G^n.*

Proof. By Proposition 13 $J_G(G[X]) = 1$ for any non-abelian torsion-free hyperbolic group G. So the desired result follows from Corollary 17.

It follows also from Corollary 17 that if G satisfies a non-trivial identity, or, more generally, a G-identity, then $G[X]$ is never the coordinate group of an algebraic set in G^n.

Finally, on combining Theorem B2 with Proposition 21 we obtain the following proposition.

PROPOSITION 22. *Let H be a G-equationally Noetherian G-group. Then for every algebraic set $Y \subseteq H^n$ the coordinate group $\Gamma(Y)$ is G-equationally Noetherian.*

5.2. *Coordinate Groups of Irreducible Algebraic Sets*

In this section we give several useful characterizations of irreducible algebraic sets in terms of their ideals and also their coordinate groups.

THEOREM D2. *Let H be a G-equationally Noetherian G-domain and Y be an algebraic set in H^n. Then the following conditions are equivalent*:

1. *Y is irreducible*;
2. *$I_H(Y)$ is a prime ideal in $G[X]$*;

3. $\Gamma(Y)$ *is a G-equationally Noetherian G-domain*;
4. $\Gamma(Y)$ *is G-discriminated by H*.

Proof. The equivalence of 1. and 2. follows from Propositions 18 and 19 (Section 4.3).

The equivalence of 2. and 3. is a consequence of the definition of a prime ideal and Proposition 22 (Section 5.1).

Condition 4. is equivalent to 1 by Proposition 21 (Section 5.1), Theorem C1 (Section 2.3), and Theorem B2 (Section 2.2).

Theorem D2 provides a useful way of proving irreducibility of algebraic sets, as we see from

PROPOSITION 23. *Let G be an equationally Noetherian non-abelian torsion-free hyperbolic group. If a finitely generated group H given by the presentation*

$$H = \langle x_1, \dots, x_n; S \rangle$$

(*so S is a subset of the free group, freely generated by X*) *is discriminated by G, then $V_G(S)$ is an irreducible algebraic set.*

Proof. Let $Q = \mathrm{gp}_{G[X]}(S)$ be the normal closure of S in $G[X]$. Then

$$G[X]/Q \simeq G * H.$$

We proved in [BMR2] that, under the above assumptions, the free product $G * H$ is G-discriminated by G. It follows that $Q = \mathrm{Rad}_G(Q)$ and hence that $G[X]/Q$ is the coordinate group of the algebraic set $V_G(S)$. So, by Theorem D2 this algebraic set $V_G(S)$ is irreducible.

There are numerous examples of groups that are discriminated by free groups in [BB] and [BG]. In [BMR1] and [BMR2] we described several constructions which provide groups discriminated by a given torsion-free hyperbolic group. Here we mention a few typical examples. The orientable surface groups

$$\langle x_1, y_1, \dots, x_n, y_n; [x_1, y_1] \cdots [x_n, y_n] = 1 \rangle$$

are discriminated by a free group, whenever $n \geq 1$, while the non-orientable surface groups

$$\langle x_1, \dots, x_n; x_1^2 \cdots x_n^2 = 1 \rangle$$

are discriminated by a free group whenever $n > 3$. It follows that all of these groups are separated by every non-abelian torsion-free hyperbolic group G, since every non-abelian, torsion-free hyperbolic group contains a non-abelian free subgroup. These results together with Proposition 23 imply

PROPOSITION 24. *Let* $S = \{[x_1, y_1] \cdots [x_n, y_n]\}$ *or let* $S = \{x_1^2 \cdots x_n^2\}$ $(n \neq 3)$. *Then for any equationally Noetherian non-abelian torsion-free hyperbolic group* G *the algebraic set* $V_G(S)$ *is irreducible.*

Proof. The orientable case and the non-orientable case for $n > 3$ both follow immediately from the discussion above. So we are left to consider only the cases when S is either $\{x_1^2\}$, or $\{x_1^2 x_2^2\}$.

If $S = \{x_1^2\}$, it is easy to see that

$$\mathrm{Rad}_G(S) = \mathrm{gp}_{G[X]}(x_1),$$

and consequently $G[X]/\mathrm{Rad}_G(S) \simeq G$. Hence $V_G(S)$ is irreducible by Theorem D2.

Similarly, if $S = \{x_1^2 x_2^2\}$, then

$$\mathrm{Rad}_G(S) = \mathrm{gp}_{G[X]}\left(x_1 x_2^{-1}\right).$$

Indeed, G is commutative transitive, therefore for any solution $x_1 = u$, $x_2 = v$ of $S = 1$ in G the elements u and v commute, therefore $(uv)^2 = 1$ and hence $uv = 1$. In this event,

$$G[X]/\mathrm{Rad}_G(S) \simeq G * \langle x_1 \rangle$$

is G-discriminated by G and the desired conclusion follows from Theorem D2.

Notice, that if the group G in Proposition 24 is free, then the conclusion also holds in the non-orientable case with $n = 3$. This follows from a result of Schutzenberger [SM] which states that if $x_1 = u$, $x_2 = v$, $x_3 = w$ is a solution of the equation $x_1^2 x_2^2 x_3^2 = 1$ in a free group F, then the elements u, v, w all commute. Hence

$$\mathrm{Rad}_F\left(x_1^2 x_2^2 x_3^2\right) = \mathrm{gp}_{F[X]}(x_1 x_2 x_3),$$

and the group

$$F[X]/\mathrm{Rad}_F\left(x_1^2 x_2^2 x_3^2\right) \simeq F * F(x_1, x_2)$$

is discriminated by F.

In the event that $G = H$ we can add one more equivalent condition to Theorem D2, which establishes a surprising relationship between coordinate groups of irreducible algebraic sets over G and finitely generated models of the universal theory of the group G.

THEOREM D3. *Let G be an equationally Noetherian domain and Y be an algebraic set in G^n. Then the following conditions are equivalent*:

1. *Y is irreducible*;
2. *$\Gamma(Y)$ is G-universally equivalent to G.*

Moreover, any finitely generated G-group which is G-universally equivalent to G is the coordinate group of some irreducible algebraic set over G.

Proof. The equivalence of 1. and 2. follows from Theorem D2 above and Theorem C2 (Section 2.3).

To prove, under either of these conditions, that any finitely generated G-group which is G-universally equivalent to G is the coordinate group of some irreducible algebraic set over G, consider a finitely generated G-group H which is G-universally equivalent to G. Then by Theorem C2, H is G-discriminated by G. Hence by Corollary 17 from Section 5.1 the group H is the coordinate group of some algebraic set over G. So the desired conclusion follows from the equivalence of 1. and 2. above.

5.3. *Decomposition Theorems*

THEOREM F1. *Let H be a G-equationally Noetherian G-domain. If Y is any algebraic set in H^n, then the coordinate group $\Gamma(Y)$ is a subgroup of a direct product of finitely many G-groups, each of which is G-discriminated by H.*

The proof follows from Proposition 12 and Theorem 5 of Section 3.6. We come now to the proof of Theorem F2.

THEOREM F2. *Let H be a non-abelian equationally Noetherian torsion-free hyperbolic group. Then every finitely generated group E which is separated by H is a subdirect product of finitely many finitely generated groups, each of which is discriminated by H.*

Proof. Let E be a finitely generated group that is separated by a non-abelian equationally Noetherian torsion-free hyperbolic group H. Let

$$E = \langle x_1, \ldots, x_n; f_1, f_2, \ldots \rangle$$

be a presentation of E on finitely many generators $x_1, \ldots, x_n$ and possibly infinitely many defining relators $f_1, f_2, \ldots$. Then we can think of the set $S = \{f_1, f_2, \ldots\}$ of defining relators of E as a subset of the free H-group $H[X]$. Observe that if $\mathrm{gp}_F(S)$ is the normal closure of S in $F = \langle x_1, \ldots, x_n \rangle$, then

$$I(V_H(S)) \cap F = \mathrm{gp}_F(S).$$

Indeed, if $f \in F$ and $f \notin \mathrm{gp}_F(S)$, then there is a homomorphism ϕ from F into H, which factors through E (i.e., $\mathrm{gp}_F(S)$ is in the kernel of ϕ) such that $\phi(f) \neq 1$. This homomorphism ϕ can be extended to a H-homomorphism, again denoted by ϕ, from $H[X]$ into the H-group H. If we put $\phi(x_i) = a_i$ $(i = 1, \ldots, n)$, then f does not vanish at $a = (a_1, \ldots, a_n)$ although $a = (a_1, \ldots, a_n) \in V_H(S)$. Hence $f \notin I(V_H(S))$. Therefore E embeds in $\Gamma(Y)$, where $Y = V_H(S)$. But H is an equationally Noetherian domain. So, by Theorem F1, we find that E is embeddable in a direct product of finitely many finitely generated H-groups, each of which is H-discriminated by H. Since every hyperbolic group, in particular H, is finitely generated it follows that every finitely generated H-groups is finitely generated as an abstract group.

COROLLARY 19. *Every finitely generated group that is separated by a free group is a subdirect product of finitely many finitely generated groups which are discriminated by free groups.*

6. THE NULLSTELLENSATZ

Hilbert's Nullstellensatz holds over an algebraically closed field K and it has various equivalent formulations. One of them asserts that every proper ideal in the polynomial ring $K[X]$ has a root in K. A similar result holds for G-groups.

THEOREM G1. *Let H be a G-algebraically closed G-group. Then every ideal in $G[X]$ which is the normal closure of a finite set, has a root in H^n.*

Proof. Let Q be an ideal in $G[X]$ which is the normal closure of the finite set S. Then $G[X]/Q$ is a G-group and S has a root in H. Since G is algebraically closed, S has a root in G.

One of the consequences of Theorem G1 is the following result which, as in the classical case, shows that there exists a one-to-one correspondence between finitely generated ideals in $G[X]$ and points in the affine space G^n.

THEOREM 6. *Let G be an algebraically closed group. Then an ideal Q in $G[X]$, which is the normal closure in $G[X]$ of a finite set, is maximal if and only if it is of the form $I(a)$, where $a \in G^n$.*

Proof. Suppose that Q is a maximal ideal in $G[X]$ and that Q is the normal closure of a finite set. Since Q has a root a in G^n, $Q = I(a)$. The converse follows on appealing to Lemma 18 (Section 4.1).

Another classical form of the Nullstellensatz describes the closed ideals I in the ring of polynomials $K[X]$ over an algebraically closed field K as radical ideals. We recall that an ideal I of $K[X]$ is a radical ideal if $f \in I$ whenever $f^n \in I$ ($f \in K[X]$). A similar result holds also for G-algebraically closed groups. In order to formulate it, we introduce the following definition.

DEFINITION 23. Let H be a G-group and let S be a subset of $G[X]$. Then we say that S satisfies the Nullstellensatz over H if

$$I(V_H(S)) = \mathrm{gp}_{G[X]}(S).$$

In particular, an ideal Q in $G[X]$ satisfies the Nullstellensatz if and only if Q is H-radical.

THEOREM G2. *Let H be a G-group and suppose that H is G-algebraically closed. Then every finite subset S of $G[x_1, \dots, x_n]$ with $V_H(S) \neq \emptyset$, satisfies the Nullstellensatz.*

Proof. Observe that if $Q = \mathrm{gp}_{G[X]}(S)$ in $G[X]$, then Q is an ideal of $G[X]$ since, by hypothesis, $V_H(S) \neq \emptyset$. Consequently the quotient group $K = G[X]/Q$ is a G-group.

Now suppose that $S = \{f_1, \dots, f_k\}$ and that $f \notin Q$. Let $H' = H *_G K$ be the free product of H and K amalgamating G. Then H' is a G-group and if $b_i = x_i Q$, we find that $f_j(b_1, \dots, b_n) = 1$ for $j = 1, \dots, k$ but that $f(b_1, \dots, b_n) \neq 1$. Since H is G-algebraically closed and is a G-subgroup of H', there exist elements $a_1, \dots, a_n$ in H such that $f_j(a_1, \dots, a_n) = 1$ for $j = 1, \dots, k$ but that $f(a_1, \dots, a_n) \neq 1$. Put $v = (a_1, \dots, a_n) \in H^n$. Then $v \in V_H(S)$ and $f(v) \neq 1$. Therefore $f \notin I(V_H(S))$. It follows that $I(V_H(S)) \subseteq Q$ and hence that $I(V_H(S)) = Q$.

The next simple but useful result follows immediately from Proposition 14 (Section 4.2).

PROPOSITION 25. *Suppose that H is a G-group and that $V_H(S) \neq \emptyset$, where S is a subset of $G[x_1, \dots, x_n]$. Then S satisfies the Nullstellensatz over H if and only if $G[X]/\mathrm{gp}_{G[X]}(S)$ is G-separated by H.*

The problem of the description of systems of equations which satisfy the Nullstellensatz, for example, over a non-abelian free group is, in general, a difficult one. Here we discuss only the standard quadratic equations, without coefficients. The orientable one of genus n takes the form

$$[x_1, y_1] \cdots [x_n, y_n] = 1,$$

while the non-orientable one of genus n takes the form

$$x_1^2 \cdots x_n^2 = 1.$$

THEOREM 7. *Let* $S = [x_1, y_1] \cdots [x_n, y_n]$ *or* $S = x_1^2 \cdots x_n^2$ (*in the latter case* $n > 3$). *Then* S *satisfies the Nullstellensatz over any torsion-free non-abelian hyperbolic group* G.

Proof. Let G be a torsion-free non-abelian hyperbolic group. Let Q be the normal closure $\mathrm{gp}_{G[X]}(S)$ of S in $G[X]$. Then

$$G[X]/Q \simeq G * F/\mathrm{gp}_F(S),$$

where here F is the free group on X. We have proved in Section 5.3 that the group $G * F/\mathrm{gp}_F(S)$ is G-discriminated by G. Consequently, S satisfies the Nullstellensatz over G, as desired.

Notice that in the case of a non-abelian free group F, Kharlampovich and Myasnikov [KM] describe the radical of an arbitrary quadratic equation (with coefficients) over F; in particular, they give a description of the quadratic equations over F that satisfy the Nullstellensatz.

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