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Stochastic modelling of the individuals's community with their transformation and interaction

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Abstract

The paper is concerned with the stochastic model of individuals's community with their transformation and interaction. The model is formalized in terms of Markov's random process with special state's space. The simulation algorithm is worked out and tested. The stochastic analog May–Anderson's model for the spread of epidemic with immunity acquisition is studied. For probabilistic model of isolated population with seasonal reproduction and competition a mean time before degeneration is analyzed.

1 Brief formulation of the model

We consider the stochastic model of individuals's community with their transformation and interaction. The model has the following assumptions. Every individual at time $t \geq 0$ belongs to one of the classes $A_1, \ldots, A_n$, $n \in \mathbf{N}$. At time $t = 0$ every class A_i, $i = 1, \ldots, n$, contains $x_i^\circ \geq 0$ primary individuals. Every primary individual x from class A_i at time $t = \ell_x$ suffers transformation, where $\ell_x > 0$ is random variable with distribution L_i°. If $t > 0$ is the time of the appearing of individual x in class A_i, then $t + \ell_x$ is the time of its transformation, where $\ell_x > 0$ is random variable with distribution L_i. As a result of the transformation the collections $\eta_{i1}^x, \ldots, \eta_{in}^x$ and $\xi_{i1}^x, \ldots, \xi_{in}^x$ of individuals respectively disappear and appear in classes $A_1, \ldots, A_n$. We suppose that $\{\ell_x\}_x$ are mutually independent, L_i° and L_i, $i = 1, \ldots, n$, are fixed distributions, $\{\eta_i^x = (\eta_{i1}^x, \ldots, \eta_{in}^x)\}_{x,\,i}$ and $\{\xi_i^x = (\xi_{i1}^x, \ldots, \xi_{in}^x)\}_{x,\,i}$ are mutually independent random vectors with fixed distributions. Individuals of the classes $A_1, \ldots, A_n$ can take part in the interactions of m different types $I_1, \ldots, I_m$. If we suppose that for $t \in (t_1; t_2)$ individuals's transformations from all classes $A_1, \ldots, A_n$ don't take place, then in the time $(t; t + \Delta) \subset (t_1; t_2)$ the individuals interact by type I_k, $k = 1, \ldots, m$, with probability $\lambda_k(x(t))\Delta + o(\Delta)$, interactions of several such types take place only with probability $o(\Delta)$, and probability of absence of interaction equals $1 - \sum_k \lambda_k(x(t))\Delta + o(\Delta)$, $\Delta \to 0$. The symbol $x(t) = (x_1(t), \ldots, x_n(t))$ means the numbers of individuals belonging to classes $A_1, \ldots, A_n$ at time $t \geq 0$, $\lambda_k = \lambda_k(x(t))$ are nonnegative functions of n-dimensional vectors with nonnegative integer coordinates. As a result of the realization I_k at time t the collections $\gamma_{k1}^t, \ldots, \gamma_{kn}^t$ and $\beta_{k1}^t, \ldots, \beta_{kn}^t$ of individuals respectively disappear and appear in classes $A_1, \ldots, A_n$. $\{(\gamma_{k1}^t, \ldots, \gamma_{kn}^t)\}_{t,\,k}$ and $\{(\beta_{k1}^t, \ldots, \beta_{kn}^t)\}_{t,\,k}$ are mutually independent random n-dimensional vectors with nonnegative integer coordinates having constant distributions.

Above-mentioned assumptions allows us to formalize the model in terms of Markov's random process with special state's space and to construct the formulas describing the rules of state's change [1,2]. The simulation algorithm was worked out and tested. The algorithm and present version of computer program allows to operate with about million individuals population. The mean value of the algorithm's run time depends on the number of individuals according to the logarithmic law. More detailed description of the model one can see in [3].

The model was used to simulate individuals's community of various types. Here we represent the model of epidemic process and the model of population with seasonal reproduction and competition.

2 Stochastic model of epidemic process

Consider the stochastic analog May–Anderson's model for the spread of epidemic with immunity acquisition [4]. Suppose that we are dealing with an individual's community of three classes: A_1 — susceptible , A_2 — infective and A_3 — immune individuals. Contact of susceptible and infective individuals leads to appearance of new infective individuals. Infectious individual is not isolated, and may die or recover with immunity acquisition. Eventually immunity losts and individual becomes susceptible to disable. All individuals belonging to community take part in reproduction of new susceptible individuals. Moreover, all individuals may perish because of external factors.

Assume that individuals of class A_1 don't suffer transformation, setting that $L_1^\circ(-\infty;t) = L_1(-\infty;t) = 0$ for all $t \geq 0$. Individual x of class A_2 with probability $p = \delta/(\alpha+\delta)$ transforms into individual of class A_3 or with probability $1-p$ perishes. Accordingly, $\mathbf{P}\{\eta_2^x = (0,0,0)\} = 1$, $\mathbf{P}\{\xi_2^x = (0,0,1)\} = p$, $\mathbf{P}\{\xi_2^x = (0,0,0)\} = 1-p$. Duration of disease (period before transformation of individual of class A_2) describes by distributions L_2° and L_2. Individual x of class A_3 with probability $p = 1$ transforms into individual of class A_1, i.e. $\mathbf{P}\{\eta_3^x = (0,0,0)\} = 1$, $\mathbf{P}\{\xi_3^x = (1,0,0)\} = 1$. Duration of immunity conservation (period before transformation of individual of class A_3) describes by distributions L_3° and L_3. Denote by $x_1 = x_1(t)$, $x_2 = x_2(t)$, $x_3 = x_3(t)$ the sizes of classes.

Individuals take part in following interactions. Interactions I_1 lead to the reproduction of susceptible individuals:

$$\lambda_1(x_1,x_2,x_3) = a\,(x_1+x_2+x_3),$$
$$\mathbf{P}\{\gamma_1^t = (0,0,0)\} = 1, \quad \mathbf{P}\{\beta_1^t = (1,0,0)\} = 1.$$

Interactions I_2 describe the contact between susceptible and infective individuals:

$$\lambda_2(x_1,x_2,x_3) = \beta\,x_1x_2, \quad \mathbf{P}\{\gamma_2^t = (1,0,0)\} = 1, \quad \mathbf{P}\{\beta_2^t = (0,1,0)\} = 1.$$

Interactions I_3, I_4, I_5 are responsible for perish of individuals because of external factors:

$$\lambda_i(x_1,x_2,x_3) = b\,x_i,$$
$$\mathbf{P}\{\gamma_3^t = (1,0,0)\} = \mathbf{P}\{\gamma_4^t = (0,1,0)\} = \mathbf{P}\{\gamma_5^t = (0,0,1)\} = 1,$$
$$\mathbf{P}\{\beta_i^t = (0,0,0)\} = 1, \quad i = 3,4,5.$$

We used the computer model to study the behavior of mathematical expectations $M_1(t)$, $M_2(t)$, $M_3(t)$ of $x_1(t)$, $x_2(t)$, $x_3(t)$ in case when finite $\lim\limits_{t\to+\infty} M_i(t) = M_i^*$, $i = 1,2,3$, exist. For estimation of mathematical expectations we used standard statistical formulas under confidence probability 0.95 and sample size $N = 1000$. The model's parameters are: $a = 5$, $b = 1$, $\alpha = 8$, $\beta = 0.1$, $\gamma = 1$, $\delta = 1$. Initial size of classes choose in accordance to binomial distribution $\mathbf{P}\{x_i^\circ = x\} = C_{N_i}^x/2^{N_i}$, $x = 1,\ldots,N_i$, $i = 1,2,3$, for $N_1 = 400$, $N_2 = 800$, $N_3 = 400$. The dependence M_i^*, $i = 1,2,3$, of distributions L_2°, L_2, L_3° and L_3 is analyzed. In table 1 we show estimations $M_1(t)$, $M_2(t)$, $M_3(t)$, if $L_2^\circ, L_2, L_3^\circ, L_3$ are given by

$$L_2^\circ = L_2 = \mathrm{E}(\alpha+\delta), \quad L_3^\circ = L_3 = \mathrm{E}(\gamma), \tag{1}$$

where $\mathrm{E}(\theta)$ is exponential distribution with parameter θ.

Table 1. Behavior of $M_i(t)$, $i = 1, 2, 3$, in case (1)

t	$M_1(t)$	$M_2(t)$	$M_3(t)$
0	199.45 ± 0.61	399.62 ± 0.85	199.75 ± 0.62
1	98.51 ± 0.76	291.14 ± 2.09	166.58 ± 1.10
2	99.05 ± 0.76	247.02 ± 2.01	133.78 ± 1.08
3	99.77 ± 0.75	224.39 ± 1.95	117.62 ± 1.06
4	99.61 ± 0.76	213.13 ± 1.88	109.27 ± 1.06
5	100.19 ± 0.78	207.09 ± 1.88	105.32 ± 1.02
6	100.46 ± 0.80	202.74 ± 1.83	102.81 ± 0.96
7	100.42 ± 0.80	202.98 ± 1.83	101.71 ± 0.94
8	99.64 ± 0.81	200.96 ± 1.81	100.39 ± 0.92
9	100.64 ± 0.81	200.56 ± 1.76	99.97 ± 0.93

Using of exponential distributions means that the stochastic process $x(t) = (x_1(t), x_2(t), x_3(t))$, $t \geq 0$, becomes Markov's process. If we neglect of pair correlations and variances of $x_1(t)$, $x_2(t)$, $x_3(t)$, then functions $M_1(t)$, $M_2(t)$, $M_3(t)$ will satisfy determinate May–Anderson's model. For indicated parameters determinate model has globally asymptotic steady-state $(100, 200, 100)$. In table 2 estimations $M_1(t)$, $M_2(t)$, $M_3(t)$ are represented with

$$L_2^\circ = \mathrm{U}(0; 1/(\alpha + \delta)), \quad L_2 = \mathrm{H}(1/(\alpha + \delta)), \tag{2}$$

$$L_3^\circ = \mathrm{U}(0; 1/\gamma), \quad L_3 = \mathrm{H}(1/\gamma), \tag{3}$$

where $\mathrm{U}(\tau_1; \tau_2)$ is uniformly distributions in $(\tau_1; \tau_2)$, $\mathrm{H}(\tau)$ is singular distribution, which is concentrated in point τ. Parameters of distributions (1) and (2), (3) are such that the mathematical expectation of corresponding random variables coinside.

When we compare the results of tables 1 and 2, we notice that the replacement of exponential distribution (1) by distributions (2), (3) doesn't affect on existence of limits M_i^* of functions $M_i(t)$, $i = 1, 2, 3$. Also values of M_2^* and M_3^* substantially increased. The similar results were obtained for another distributions (2), (3).

Table 2. Behavior of $M_i(t)$, $i = 1, 2, 3$, in case (2), (3)

t	$M_1(t)$	$M_2(t)$	$M_3(t)$
0	199.69 ± 0.63	400.38 ± 0.87	200.26 ± 0.64
1	169.29 ± 0.87	436.15 ± 2.80	251.95 ± 1.52
2	95.06 ± 0.69	394.90 ± 3.01	248.80 ± 1.81
3	95.16 ± 0.73	368.70 ± 3.11	226.41 ± 1.84
4	95.67 ± 0.68	352.08 ± 2.98	212.83 ± 1.73
5	95.34 ± 0.69	341.71 ± 3.02	207.03 ± 1.75
6	94.80 ± 0.70	335.73 ± 2.90	201.24 ± 1.67
7	95.54 ± 0.71	329.12 ± 2.87	198.69 ± 1.68
8	95.71 ± 0.71	328.38 ± 2.88	196.90 ± 1.70
9	95.37 ± 0.74	329.19 ± 2.87	197.01 ± 1.66

3 Stochastic model of isolated population

In [5] the probabilistic model of isolated population with seasonal reproduction offered. The model deals with the community in which every individual suffers transformation in fixed moments of time (for example, once a year). Transformation means the reproduction of individual which doesn't take place in another time by virtue of biological features. During the periods between transformations individuals may perish as a result of selflimitation and competition.

Suppose that all individuals are included in one class A_1 and their reproduction takes place in fixed moments $t_k = k\,h$, where $k = 0, 1, \ldots$, $h = \text{const} > 0$. Therefore, $L_1^\circ = L_1 = \mathrm{H}(h)$. Every individual x transforms into i individuals with probability p_i: $\mathbf{P}\{\eta_1^x = 0\} = 1$, $\mathbf{P}\{\xi_1^x = i\} = p_i$, $i = 0, 1, \ldots$. Single type of interactions I_1 describes individuals's death because of selflimitation and competition:

$$\lambda_1(x) = \mu\, x^2, \quad \mu = \text{const} > 0, \quad \mathbf{P}\{\gamma_1^t = 1\} = 1, \quad \mathbf{P}\{\beta_1^t = 0\} = 1.$$

For such choice of function $\lambda_1(x)$ described random process $x_1(t)$ degenerates with probability 1 [5].

We analyzed dependence of mean time T before degeneration of the process $x_1(t)$ as function of parameter μ with fixed $x_1^\circ = 1$, $h = 1$, $p_1 = 0.5$, $p_2 = 0.5$. The choice of probabilities p_1, p_2 corresponds to mathematical expectation of offspring $m = 1.5$. For estimation T we used standard statistical formulas under confidence probability 0.95 and sample size $N = 1000$. The result of calculations we show in table 3.

Table 3. Estimation of mean time T

i	μ_i	T_i	i	μ_i	T_i
1	0.045	14942.41 ± 1019.06	19	0.35	4.07 ± 0.283
2	0.05	5461.16 ± 369.03	20	0.40	3.29 ± 0.235
3	0.06	1294.69 ± 92.43	21	0.45	2.77 ± 0.204
4	0.07	408.53 ± 29.64	22	0.50	2.35 ± 0.167
5	0.08	221.08 ± 16.09	23	0.60	1.92 ± 0.131
6	0.09	114.17 ± 8.40	24	0.70	1.62 ± 0.115
7	0.10	70.74 ± 5.25	25	0.80	1.41 ± 0.095
8	0.11	50.04 ± 3.74	26	0.90	1.25 ± 0.082
9	0.12	40.52 ± 2.89	27	1.00	1.11 ± 0.069
10	0.13	32.05 ± 2.44	28	1.50	0.69 ± 0.046
11	0.14	25.12 ± 1.79	29	2.00	0.50 ± 0.034
12	0.15	21.48 ± 1.65	30	2.50	0.39 ± 0.025
13	0.16	18.17 ± 1.34	31	3.00	0.33 ± 0.020
14	0.17	14.97 ± 1.04	32	3.50	0.28 ± 0.016
15	0.18	12.79 ± 0.93	33	4.00	0.25 ± 0.015
16	0.20	10.96 ± 0.80	34	4.50	0.23 ± 0.014
17	0.25	7.29 ± 0.51	35	5.00	0.20 ± 0.012
18	0.30	5.38 ± 0.38			

To fit the data from table 3 we used function $\hat{T}(\mu)$, which minimized the sum of squares of relative differences $\Delta(\hat{T}) = \sum\limits_i (1 - \hat{T}(\mu_i)/T_i)^2$.

Table 4. The results of fitting the data from table 3

	Function $\hat{T}$	$\Delta(\hat{T})$	Residual variance
1	$\hat{T}(\mu) = B\,\mu^{-p}$	6.8948	0.2089
2	$\hat{T}(\mu) = B\,\mu^{-p}\,e^{-C\mu}$	4.9344	0.1542
3	$\hat{T}(\mu) = B\,\mu^{-p}\,e^{C/\mu}$	0.8035	0.0251
4	$\hat{T}(\mu) = B\,e^{-D\mu^p + C\,\mu^{-q}}$	0.0474	$1.579 \cdot 10^{-3}$
5	$\hat{T}(\mu) = B\,\mu^{-p}\,e^{C\mu^{-q}}$	0.0469	$1.512 \cdot 10^{-3}$

From table 4 we see that the most acceptable function is $\hat{T}(\mu) = B\,\mu^{-p}\,e^{C\mu^{-q}}$ with parameters $B = 1.0172$, $C = 6.1933 \cdot 10^{-2}$, $p = 1.0287$, $q = 1.4970$. This function allows to predict mean time before community's degeneration for given parameters and coefficient μ of selflimitation.

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