Complexity of Buffer Capacity Allocation Problems
for Production Lines with Unreliable Machines[1]

A. Dolgui*, A. Eremeev**, M.Y. Kovalyov*** and V. Sigaev**

*Ecole Nationale Supérieure des Mines
FAYOL-EMSE, CNRS UMR 6158, LIMOS
158, cours Fauriel, 42023 Saint-Etienne cedex 2, France
dolgui@emse.fr

** Omsk Branch of Sobolev Institute of Mathematics
13, Pevtsov str., 644099, Omsk, Russia
eremeev@ofim.oscsbras.ru

*** United Institute of Informatics Problems
National Academy of Sciences of Belarus
Surganova 6, 220012 Minsk, Belarus
kovalyov_my@newman.bas-net.by

Abstract. Buffer capacity allocation problems for flow-line manufacturing systems with unreliable machines are studied. These problems arise in a wide range of manufacturing systems and concern determining buffer capacities with respect to a given optimality criterion which can depend on the average production rate of the line, buffer cost, inventory cost, etc. Here, this problem is proven to be NP-hard for a tandem production line and oracle representation of the revenue and cost functions, and NP-hard for a series-parallel line and stepwise revenue function.

Key words: Flow-line, Unreliable machines, Buffer allocation, Optimization, Computational complexity.

1 Introduction

Buffer capacity allocation problems arise in a wide range of manufacturing systems, such as transfer lines, flexible manufacturing or robotic assembly systems which are flow lines. Buffers separate any two consecutive machines. The parts are accumulated in

---

[1] The final publication is available at www.springerlink.com

a buffer when the machines downstream are less productive than machines upstream. Assume that machines can breakdown. When a breakdown occurs, the corresponding machine is not used in production for a random repair time which is independent on the number of failed machines. It is assumed that there is a sufficient number of raw parts at the input of the system and these parts are always available. The completed parts depart from the system immediately. The performance of the flow-line is measured in terms of the average production rate, i.e., the steady state average number of parts produced per unit of time.

In the literature, there are two types of publications. The first concerns only evaluation of the line performance for a given size of buffers. In the second, the buffer sizes are optimized. For example, (Dallery and Gershwin, 1992), (Gershwin, 1993), (Heavey et al., 1993), (Meerkov and Li, 2008), and (Tan and Gershwin, 2009) proposed models to evaluate the performance of lines with unreliable machines and fixed sizes of buffers. Markov models and aggregation or decomposition techniques are often used to calculate steady state throughput or other performance indicators for these lines provided that the buffer capacities are given. Based on these models for performance analysis, in, e.g., (Smith and Daskalaki, 1988), (So, 1997), (Gershwin and Schor, 2000), and (Shi and Gershwin, 2009), the optimization for buffer capacity allocation was considered with respect to diverse optimality criteria and for different types of lines.

In the present paper, we assume that a machine can be in an *operational* state or *under repair*. An operational machine may be *blocked* and temporarily stopped in case if there is no room in the downstream buffer. It may also be *starved* if there are no parts to process in the upstream buffer. Otherwise operational machines are working. In what follows, m denotes the number of machines in a production line. A working machine i, $i=1,\ldots,m$, is assumed to have a constant cycle time C_i and, then, the average production rate $u_i=1/C_i$.

It is supposed that machines break down only when they are working. The times to fail and times to repair for each machine are assumed to be mutually independent and exponentially distributed random values. Let T_b^i denote the average time to fail, and let $\lambda_i=1/T_b^i$ be the failure rate for working machine i, $i=1,\ldots,m$. Similarly, let T_r^i and $\mu_i=1/$

T_r^i denote respectively the time to repair and the repair rate for machine i. Under the above mentioned assumptions the system has the steady state mode (see e.g. (Sevast'yanov, 1962)), and performance of the system in this state is important for applications.

Let the buffers in the system be denoted by $B_1,\ldots,B_N$ and let h_j be the capacity of buffer B_j, which is to be decided. Denote the vector of decision variables as $H=(h_1, h_2,\ldots, h_N)\in Z_+^N$, where Z_+ is the set of non-negative integers. The most commonly used optimization criteria are:

- Average production rate (steady state throughput) $V(H)$;
- Total buffer capacity $B(H)=h_1+h_2+\ldots+h_N$ or buffer cost $C(H)$ linear in H;
- Average steady state inventory cost $Q(H)=c_1q_1(H)+\ldots+c_Nq_N(H)$, where $q_j(H)$ is the average steady state number of parts in buffer B_j, for $j=1,\ldots,N$.

Let us introduce the following additional notation:

T_{am} amortization time of the line (line life);

$R(V)$ revenue related to the production rate V;

$J(H)$ cost of buffer configuration H;

d_j maximal admissible capacity of buffer B_j, $j=1,\ldots,N$.

$R(V)$ and $J(H)$ are assumed to be given monotone non-decreasing real-valued functions. The cost function $J(H)$ may be non-linear to model some standard buffer capacities or penalize solutions where the total capacity of the buffers exceeds an upper bound. A non-linear revenue function $R(V)$ can model the law of diminishing returns, for example, it can reflect the effect of overproduction by switching from strictly increasing to constant at a certain threshold. A stepwise revenue function can be used to model zero revenue in case of an unacceptably low average production rate (see e.g. Section 3). Consider the following criterion:

$$Max\ \varphi(H)=T_{am}\,R(V(H)) - J(H). \qquad (1)$$

Function $\varphi(H)$ has to be maximized, subject to the constraints $h_1 \leq d_1,\ h_2 \leq d_2,\ldots,\ h_N \leq d_N.$

In our previous work, we proposed several metaheuristics (Dolgui et al., 2002; 2007) for some problems of this type. In these metaheuristics, we used a two-machine one-buffer Markov model (Levin & Pasjko, 1969; Dubois & Forestier, 1982; Coillard & Proth, 1984; and Dolgui, 1993) - see some elements necessary here in Appendix - and an aggregation algorithm (Dolgui, 1993; Dolgui and Svirin, 1995), which is similar to the Terracol and David (1987) techniques to evaluate the average production rate of each tentative buffer allocation decision for the more general case of series-parallel lines with more than two machines. This aggregation approach appears to be sufficiently rapid for the evaluation of tentative buffer allocations within the optimization algorithms.

In our previous publications, two configurations of flow lines were considered: tandem lines where machines are in series and lines with series-parallel machines. This paper deals with the computational complexity of buffer allocation problems for these two line configurations.

2. Tandem production lines

A tandem production line (see Figure 1) consists of machines in series. Parts move from one machine to the next by a transfer mechanism. There is a buffer between each two successive machines, so $N = m\text{-}1$. The input data of a problem instance consists of λ_i, μ_i, u_i; $i=1,\ldots,m$; $d_j, j=1,\ldots,N$; T_{am}, and functions $R(V)$ and $J(H)$.
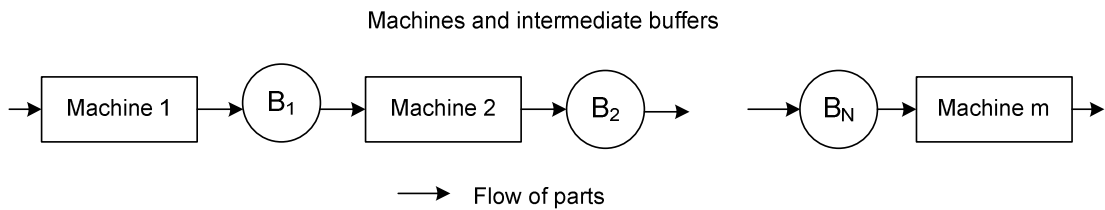
Machines and intermediate buffers

Figure 1. Tandem production line

Consider the case of $N = 1$ in which there are two machines separated by a single buffer. For this case, there exist closed-form expressions for $V(h_1)$ (see e.g. Coillard and Proth (1984) and Dolgui, (1993)), so it is appropriate to assume that $V(h_1)$ may be computed in polynomial time. In contrast, the functions $R(V)$ and $J(h_1)$ are assumed to be arbitrary, for example, they can be represented by oracles. This means that there is an algorithm,

formula, or table to compute such a function. In addition, the manner of this computation can be arbitrary and is independent of the input. A similar problem formulation has an exponential *black box complexity* as it is proved in (Dolgui, Eremeev, Sigaev, 2007). To show the NP-hardness of this special case of our problem, we will construct a reduction similar to that of Cheng and Kovalyov (2002).

Proposition 1. *The buffer capacity allocation problem is NP-hard for N=1, given that the functions $R(h_1)=R(V(h_1))$ and $J(h_1)$ are represented by oracles.*

Proof. Let us define $d=d_1$ and $h=h_1$ for shortness. We will reduce the well-known NP-complete Partition problem (Garey and Johnson, 1979) to the buffer capacity allocation problem with $N=1$. The recognition version of Partition can be formulated as follows: given n positive integers $a_1,a_2,...,a_n$, is there a Boolean vector $(y_1,y_2,...,y_n) \in \{0,1\}^n$ such

that $\sum_{j=1}^{n} a_j y_j = \frac{1}{2}\sum_{j=1}^{n} a_j$?

Given any instance of Partition, construct an instance of the buffer capacity allocation problem with $N=1$, $T_{am}=1$ and $\lambda_i=\mu_i=u_i=1$, $i=1,2$. Then $V(h)$ is a strictly increasing rational function of h, defined at every point $h \in [0,d]$, as it follows from the closed-form expressions presented in (Dubois and Forestier, 1982; Coillard and Proth, 1984; Dolgui, 1993). The parameter d is set equal to 2^n-1. Let $y(h)$ be the binary representation of integer h using n bits, so $y(h)$ is a one-to-one mapping from $\{0,1,...,d\}$ to $\{0,1\}^n$ computable in $O(n)$ time. Then we can define oracle for the buffer cost function such that $J(h)=V(h)$, and express the oracle for the revenue function such that:

$$R(h) = \begin{cases} V(h+1), & if \ \sum_{j=1}^{n} a_j y(h)_j = \frac{1}{2}\sum_{j=1}^{n} a_j, \\ \\ V(h), & otherwise. \end{cases}$$

Note that the functions $R(h)$ and $J(h)$ in this case are computable in polynomial time with respect to the input length of the Partition problem. Moreover, due to the strict monotonicity of $V(h)$, the inequality $\varphi(h)=T_{am} R(h) - J(h) > 0$ holds if and only if $y(h)$

satisfies the equation $\sum_{j=1}^{n} a_j y(h)_j = \frac{1}{2}\sum_{j=1}^{n} a_j$. In such a case, the optimal value of the

objective function in the buffer capacity allocation problem answers the question of the Partition problem. □

In the next section, we examine NP-hardness for the case of series-parallel lines with stepwise revenue function.

3. Series-parallel line with stepwise revenue function

For series-parallel lines, the structure of a line is described by a series-parallel digraph $G=(V,E)$, where V is the set of vertices, $|V|=N+2$, $|E|=m$. Vertex v_j, $j=1,\ldots,N,$ corresponds to the intermediate buffer B_j, vertex v_0 models infinite supply of raw parts at the input of the line and vertex v_{N+1} corresponds to the output of the line, where all parts are completely processed. The arcs $e_1,\ldots,e_m$ model the machines $1,\ldots,m$. Each arc e_i, $i=1,\ldots,m,$ is directed from the vertex that models the input buffer of machine i (or vertex v_0 if the machine i receives the raw parts) to the vertex that models the output buffer of machine i (or vertex v_{N+1} if the machine produces compete parts).

A graph is called series-parallel if it can be obtained from a pair of vertices connected by an arc (v_0 and v_{N+1} in our case) with the help of the following two operations: (i) adding an arc in parallel to an existing arc; (ii) substituting an arc by a simple path where all arcs have the same direction as the original arc.

Obviously, series-parallel lines constitute a generalization of tandem lines. Figure 2 illustrates an example of a series-parallel line.
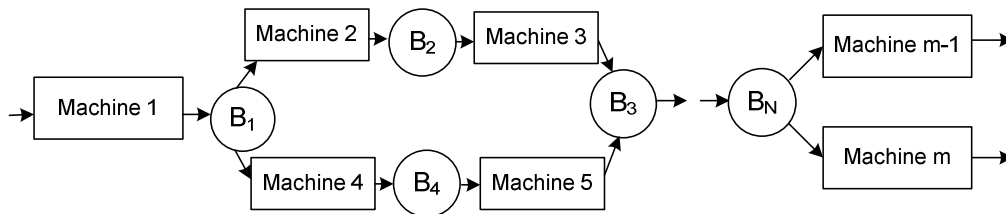

Figure 2. A line with series-parallel structure

Many authors consider the knapsack-type problems assuming that the cost function $J(H)$ is linear and that the minimum required production rate V^0 is given. Such problems may be considered in terms of our problem formulation assuming that the revenue function $R(V(H))$ is stepwise: if $V(H)$ is above a given threshold V^0, then $R(V(H))$ equals to some sufficiently large constant M, otherwise $R(V(H))=0$. This knapsack-type problem for series-parallel lines turns out to be *NP*-hard. Again, to prove this, we need to have a procedure that computes the average production rate in polynomial time. It will be sufficient for us to present such a procedure only for series-parallel lines with a *simple structure*. By system with a simple structure, we mean any line which is represented by a digraph G consisting of paths starting at vertex v_0 and ending at vertex v_{N+1}, where each path consists of 2 arcs and different paths do not have any other common vertices except for v_0 and v_{N+1}. An example of a system with simple structure is provided in Figure3.
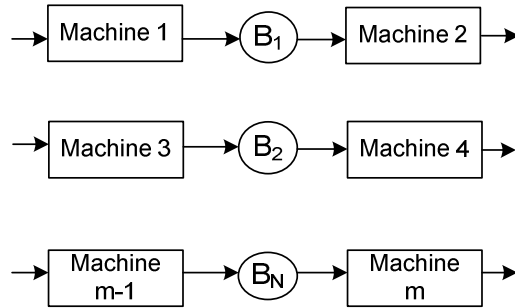

Figure 3. Example of a series-parallel line with simple structure (several two-machine tandem lines in parallel)

The input data of a problem instance consist of graph G, parameters λ_i, μ_i, u_i, $i=1,\ldots,m$, d_j, $j = 1,\ldots,N$ and T_{am}. The functions $R(V)$ and $J(H)$ are given by parameters d_j, $j = 1,\ldots,N$, V^0 and M. Equivalently, one can neglect the functions $R(V)$ and $J(H)$ and consider the problem of minimizing the function $\sum_{j=1}^{N} b_j h_j$, subject to the *constraints* $V(H) \geq V^0$, $h_1 \leq d_1$, $h_2 \leq d_2,\ldots, h_N \leq d_N$.

Proposition 2. *The problem of finding a buffer capacity allocation vector $H = (h_1,$*

$h_2,..., h_N) \in Z_+^N$ minimizing the function $\sum_{j=1}^{N} b_j h_j$, subject to the constraints $V(H) \geq V^0$,

$h_1 \leq d_1$, $h_2 \leq d_2$,..., $h_N \leq d_N$ for a series-parallel line with a simple structure and rational

weights $b_1,..., b_N$, V^0 and λ_i, μ_i, u_i, $i = 1,...,m$, is NP-hard.

Proof. See **(**Dolgui, Eremeev, Sigaev, 2007).

Let us now consider minimization of the average steady state inventory cost function
$Q(H)$. The input data of a problem instance with this criterion consist of graph G,
parameters λ_i, μ_i, u_i, $i = 1,...,m$, d_j, $j = 1,...,N$ and T_{am}. The functions $R(V)$ and $Q(H)$ are
given by parameters d_j, c_j, $j = 1,...,N$, V^0 and M. Again, one can neglect the functions

$R(V)$ and $Q(H)$ and consider the problem of minimizing the function $\sum_{j=1}^{N} c_j q_j(H)$,

subject to the *constraints* $V(H) \geq V^0$, $h_1 \leq d_1$, $h_2 \leq d_2$,..., $h_N \leq d_N$. Computational
complexity of this knapsack-type problem is considered below.

Proposition 3. *The problem of finding a buffer capacity allocation vector $H=(h_1, h_2,...,$*

$h_N) \in Z_+^N$ minimizing the function $\sum_{j=1}^{N} c_j q_j(H)$, subject to the constraints $V(H) \geq V^0$,

$h_1 \leq d_1$, $h_2 \leq d_2$,..., $h_N \leq d_N$ for a line with a simple structure and rational weights

$c_1,...,c_N$, V^0 and λ_i, μ_i, u_i, $i = 1,..., m$, is NP-hard.

Proof. Consider a special case of the problem with $m=2N$, where each path consists of
two sequential machines indexed $i,i+1$, for all odd i, $1 \leq i \leq N$. Let $j = (i+1)/2$ be the
index of the buffer between machines i and $i+1$. Assume also that $\lambda_i=2 u_i$, $\mu_i=4u_i$ for all
$i=1,...,m$, $d_1 = d_2 =...= d_N=1$ and $u_i = u_{i+1}$ for all odd i.

In a line with simple series-parallel structure considered here, all pairs of sequential
machines work independently from each other, so the steady-state throughput of such a
pair and the steady-state average number of parts in the buffer between the machines
depend only on parameters of the two machines. Let V'_j be the throughput of the two-
machine tandem line with buffer B_j, $j=1,...,N$. By means of formulas from (Dubois and

Forestier, 1982) in case $h_j=1$ we obtain: $V'_j=8u_{2j}/13$, and if $h_j=0$, we have: $V'_j=u_{2j}/2$, $j=1,\ldots,N$.

Machines within each pair have identical parameters, so $q_j(H) = 1/2\ h_j$ (see the Appendix). In the case $h_j=1$ holds $q_j(H)=1/2$, and when $h_j=0$, we have $q_j(H)=0$, $j=1,\ldots,N$. The throughput of the whole system is $V(H)=V'_1+V'_2+\ldots+V'_N$. Therefore, all the necessary system parameters are computable in polynomial time.

To show the *NP*-hardness we will again use the Partition problem, see Proposition 1.

We set $N=n$, $u_{2j}=u_{2j+1}=a_j26/3$, $c_j=2a_j$, $j=1,\ldots,N$ and $V^0=\dfrac{29}{6}\sum\limits_{j=1}^{N}a_j$. In case $h_j=1$ we

have $V'_j=a_j\cdot16/3$, $q_j(H)=a_j$ and otherwise $V'_j=a_j\cdot13/3$, $q_j(H)=0$, $j=1,\ldots,N$.. Therefore the total throughput is

$$V(H)=\sum_{j=1}^{N}V'_j = \sum_{j=1}^{N}a_jh_j + \frac{13}{3}\sum_{j=1}^{N}a_j = \sum_{j=1}^{N}a_jh_j + V^0 - \frac{1}{2}\sum_{j=1}^{N}a_j .$$

So, $H\in\{0,1\}^N$ is feasible if and only if $\sum\limits_{j=1}^{N}a_jh_j \geq \dfrac{1}{2}\sum\limits_{j=1}^{N}a_j$. It is easy to see that the

Partition problem has the affirmative answer if and only if $\sum\limits_{j=1}^{N}c_jq_j(H) = \sum\limits_{j=1}^{N}a_jh_j \leq$

$\dfrac{1}{2}\sum\limits_{j=1}^{N}a_j$. Therefore the special case of buffers capacity allocation problem considered in

this proposition is NP-hard. □

3. Conclusion

In this paper, we have proven NP-hardness of the buffer capacity allocation problem for two cases: 1) tandem production lines with oracle representation of the revenue and cost functions, and 2) series-parallel lines of simple structure with stepwise revenue function.

Acknowledgments

Appendix

In this appendix we describe the production line model used for two-machine line analysis, i.e. when the number of buffers is $N=1$ and the number of machines is $m=2$ (Dubois and Forestier, 1982). Let h be the capacity of the buffer between the machines. In this case the system states can be expressed by the triple (α_1, α_2, x), where $\alpha_i = 0$ if machine i is under repair, and $\alpha_i = 1$ if it is operational. The value $x \in [0, h]$ denotes the amount of buffer capacity used by the parts. In this discrete-continuous model for the two-machine serial system we have a set of intermediate states $\{0,1\}^2 \times] 0, h [$ and the boundary states $(\{0,1\}^2 \times \{0\}) \cup (\{0,1\}^2 \times \{h\})$. Let A_i denote the binary random variable for the state of machine i, $i=1,2$, and let X be the random variable for the amount of buffer capacity used by the parts. The probabilistic characteristics in this model are given by the probabilities of boundary states at time t:

$$P_{\alpha_1\alpha_2}(0,t)=P\{(A_1, A_2, X)=(\alpha_1, \alpha_2, 0) \text{ at time } t\},$$

$$P_{\alpha_1\alpha_2}(h,t)=P\{(A_1, A_2, X)=(\alpha_1, \alpha_2, h) \text{ at time } t\},$$

and the probability density $f_{\alpha_1\alpha_2}(x,t)$ of intermediate states: $f_{\alpha_1\alpha_2}(x,t)=\partial F_{\alpha_1\alpha_2}(x,t)/\partial x$, where $F_{\alpha'_1\alpha'_2}(x,t)$ is the probability that at moment t the system state (A_1, A_2, X) is such that $A_1=\alpha_1$, $A_2=\alpha_2$, and $X < x$. The asymptotic steady-state distributions are described by

$$P_{\alpha_1\alpha_2}(0)=\lim_{t\to\infty}P_{\alpha_1\alpha_2}(0,t), \ P_{\alpha_1\alpha_2}(h)=\lim_{t\to\infty}P_{\alpha_1\alpha_2}(h,t),$$

$$f_{\alpha_1\alpha_2}(x)=\lim_{t\to\infty}f_{\alpha_1\alpha_2}(x,t).$$

We use the asymptotic relationships obtained in (Dubois and Forestier, 1982). For the intermediate states we have:

$$0=\lambda_1 f_{10}(x)+ \lambda_2 f_{01}(x)-(\mu_1+\mu_2)f_{00}(x),$$

$$-u_2 \, \partial f_{01}(x)/\partial x=\lambda_1 f_{11}(x)+ \mu_2 f_{00}(x)-(\mu_1+\lambda_2)f_{01}(x),$$

$$u_1 \, \partial f_{10}(x)/\partial x=\lambda_2 f_{11}(x)+\mu_1 f_{00}(x)-(\lambda_1+\mu_2)f_{10}(x),$$

$$(u_1-u_2)\partial f_{11}(x)/\partial x=\mu_1 f_{01}(x)+\mu_2 f_{10}(x)-(\lambda_1+\lambda_2)f_{11}(x)$$

For the boundary states three cases should be considered: $u_1>u_2$, $u_1<u_2$ and $u_1=u_2=u$. We use only the latter case, where the following equations can be proved:

$$P_{01}(h)=P_{00}(h)= P_{10}(0)=P_{00}(0)=0,$$

$$\mu_2 P_{10}(h)= \lambda_2 P_{11}(h)+ uf_{10}(h)=(\lambda_1+\lambda_2)P_{11}(h),$$

$$\mu_1 P_{01}(h)= \lambda_1 P_{11}(0)+ uf_{01}(0)=(\lambda_1+\lambda_2)P_{11}(0),$$

$$u f_{01}(h)= \lambda_1 P_{11}(h), \ uf_{10}(0)= \lambda_2 P_{11}(0).$$

Solution of this system yields the analytic expressions for $f_{\alpha_1\alpha_2}(x)$, $P_{\alpha_1\alpha_2}(0)$, and $P_{\alpha_1\alpha_2}(h)$. Using these functions one can obtain the average production rate $V(h)$. For example in case $\lambda_1=\lambda_2=\lambda$, $\mu_1=\mu_2=\mu$ we have:

$$F_{10}(x)=kx, \ F_{00}(x)=kx\lambda/\mu, \ F_{11}(x)=kx\mu/\lambda, \ F_{01}(x)=kx,$$

$$P_{11}(h)=ku/\lambda, \ P_{10}(h)=2ku/\mu, \ P_{00}(h)= P_{01}(h)=0,$$

$$P_{11}(0)=ku/\lambda, \ P_{01}(0)=2ku/\mu, \ P_{00}(0)= P_{10}(0)=0,$$

where k is found from normalization condition $\displaystyle\sum_{\substack{\alpha_1 \in \{0,1\}, \\ \alpha_2 \in \{0,1\}}} [F_{\alpha_1\alpha_2}(h) + P_{\alpha_1\alpha_2}(h)+ P_{\alpha_1\alpha_2}(0)] = 1$.

In the complexity analysis within this paper we employ only the case where $\mu_1=\mu_2$ and $\lambda_1=\lambda_2$. Let $I=\mu_1/\lambda_1=\mu_2/\lambda_2$. Then the average production rate is given by:

$$V(h) = u \left[1+I + \cfrac{I}{1+h(1+I)\dfrac{\mu_1\mu_2}{(\mu_1 + \mu_2)u}} \right]^{-1}.$$

The value $Q(H)$ can be found by summing the expected amounts of parts (multiplied by the inventory costs) in buffers. In particular, for a two-machine tandem line as considered above, the steady state average number of parts in the intermediate buffer q is derived as follows. For any of the four combinations of the indices $\alpha_1 \in \{0,1\}$, $\alpha_2 \in \{0,1\}$ at any moment t we can define a random value $Z(\alpha_1, \alpha_2)$, assuming $Z(\alpha_1, \alpha_2) = X$, if the system is in a state (A_1, A_2, X), such that $A_1 = \alpha_1$, $A_2 = \alpha_2$; otherwise $Z(\alpha'_1, \alpha'_2) = 0$.

Asymptotically, as t tends to ∞, the distribution function for each random variable $Z = Z(\alpha_1, \alpha_2)$, $\alpha_1 \in \{0,1\}$, $\alpha_2 \in \{0,1\}$, is

$$F_Z(x) = P\{Z < x\} = P\{Z < x \ \& \ (A_1, A_2) = (\alpha_1, \alpha_2)\} + P\{Z < x \ \& \ (A_1, A_2) \neq (\alpha_1, \alpha_2)\}.$$

Note that

$$P\{Z < x \ \& \ (A_1, A_2) = (\alpha_1, \alpha_2)\} = \begin{cases} F_{\alpha_1 \alpha_2}(x), & if \ x \leq h, \\ \\ P\{(A_1, A_2) = (\alpha_1, \alpha_2)\}, & otherwise \end{cases}$$

by definition of $F_{\alpha_1 \alpha_2}(x)$, and

$$P\{Z < x \ \& \ (A_1, A_2) \neq (\alpha_1, \alpha_2)\} = \begin{cases} 0, & if \ x \leq 0, \\ \\ 1 - P\{(A_1, A_2) = (\alpha_1, \alpha_2)\}, & otherwise. \end{cases}$$

Thus, in view of properties of expectation (see e.g. (Gnedenko, 1997)), using the definitions of $F_{\alpha_1 \alpha_2}(x)$ and $P_{\alpha_1 \alpha_2}(h)$ and the fact that $F_Z(x) \equiv 1$ for $x > h$, we obtain

$$E[Z] = \int_0^\infty [1 - F_Z(x)] \, dx = \int_0^h [1 - F_Z(x)] \, dx = \int_0^h \left[1 - F_{\alpha_1 \alpha_2}(x) - 1 + P\{(A_1, A_2) = (\alpha_1, \alpha_2)\} \right] dx =$$

$$\int_0^h \left[F_{\alpha_1 \alpha_2}(h) + P_{\alpha_1 \alpha_2}(h) - F_{\alpha_1 \alpha_2}(x) \right] dx ,$$

and in total,

$$q = \sum_{\substack{\alpha_1 \in \{0,1\}, \\ \alpha_2 \in \{0,1\}}} \int_0^h \left[F_{\alpha_1 \alpha_2}(h) - F_{\alpha_1 \alpha_2}(x) \right] dx + h \cdot \sum_{\substack{\alpha_1 \in \{0,1\}, \\ \alpha_2 \in \{0,1\}}} P_{\alpha_1 \alpha_2}(h).$$

With $u_1 = u_2 = u$, $\mu_1 = \mu_2$ and $\lambda_1 = \lambda_2$ it yields $q = h/2$.

The above model is developed for two machines with a buffer in between. For a series-parallel line with more machines and buffers, the aggregation algorithm for production rate evaluation consists in applying the following two rules:

R_1: two machines in series are replaced with an equivalent machine

R_2: two machines in parallel are replaced with an equivalent machine

The parameters λ^*, μ^*, u^* of the equivalent machines are calculated from differential equations corresponding to the Markov models of two machines in series and one buffer in between (R1) or two

parallel machines (R2). More details on these equations may be found in (Dolgui, 1993; Dolgui et al. 2002, 2007).

After K-1 steps of such aggregation procedure the system reduces to a single machine with parameters λ^*, μ^*, u^* and the estimate of the overall production rate $V(H)$ is given by $u^*\mu^*/(\lambda^*+\mu^*)$.

The precision of this approximate method depends on the order in which the line is aggregated. In our study, the replacement of type R_1 is always applied where possible before the reduction of type R_2.
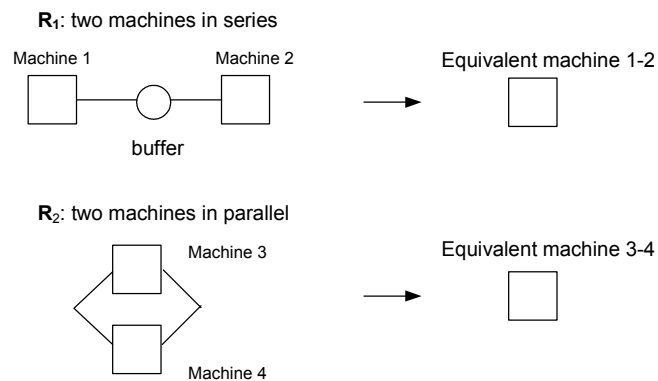

Fig. A1 Decomposition rules

In case there are several alternatives, the rule R_1 is applied to the couple of machines, separated by a buffer of the least capacity. The value $Q(H)$ is estimated by summing the expected amounts of parts (multiplied by the inventory costs) in buffers being eliminated on each step of aggregation.

References

Cheng T.C.E., Kovalyov M. Y. (2002) An unconstrained optimization problem is NP-hard given an oracle representation of its objective function: a technical note, *Computers & Operations Research* **29**, 2087–2091.

Coillard P. and Proth J.M. (1984) Effet des stocks tampons dans une fabrication en ligne, *Revue belge de Statistique, d'Informatique et de Recherche Opérationnelle* **24** (2), 3-27.

Dallery Y. and Gershwin S.B., (1992) Manufacturing flow line systems: a review of models and analytical results, *Queueing Systems* **12** (1-2), 3-94.

Dolgui A. (1993) *Analyse de performances d'un atelier de production discontinue: méthode et logiciel*, Research Report INRIA 1949, 44 pages.

Dolgui A., Eremeev A.V., Kolokolov A.A., Sigaev V.S. (2002) A genetic algorithm for allocation of buffer storage capacities in production line with unreliable machines. *Journal of Mathematical Modelling and Algorithms* **2**, 89-104.

Dolgui A., Eremeev A.V., Sigaev V.S. (2007) HBBA: hybrid algorithm for buffer allocation in tandem production lines. *Journal of Intelligent Manufacturing* **18** (3), 411-420.

Dolgui A.B. and Svirin Y.P. (1995) Models of evaluation of probabilistic productivity of automated technological complexes, *Vesti Akademii Navuk Belarusi*: *phisika-technichnie navuki*, n°1, 59-67 (in Russian).

Dubois D. and Forestier J.P. (1982) Productivité et en-cours moyens d'un ensemble de deux machines séparées par une zone de stockage, *RAIRO Automatique* **16** (2), 105-132.

Garey M.R. and Johnson D.S. (1979) *Computers and Intractability. A Guide to the theory of NP-completeness,* W.H. Freeman and Company, San Francisco.

Gershwin S.B. (1993). *Manufacturing Systems Engineering*, Prentice Hall.

Gershwin S.B. and Schor J.E. (2000) Efficient algorithms for buffer space allocation, *Annals of Operations Research* **93**, 117-144.

Gnedenko B.V. (1997) *Theory of probability,* Gordon and Breach, Amsterdam.

Heavey C., Papadopoulos H.T. and Browne J. (1993) The throughput rate of multistation unreliable production lines, *European Journal of Operational Research* **68**, 69-89.

Levin A.A. and Pasjko N.I. (1969) Calculating the output of transfer lines, *Stanki i Instrument* **8**, 8-10 (in Russian).

Li J. and Meerkov S.M. (2008) *Production Systems Engineering*, Springer.

Sevast'yanov B. A. (1962) Influence of storage bin capacity on the average standstill time of a production line, *Theory Probab. Appl,* **7 (**4), 429–455

So K.C. (1997) Optimal buffer allocation strategy for minimizing work-in-process inventory in unpaced production lines, *IIE Transactions* 29, 81-88

Shi C. and Gershwin S. B. (2009) An efficient buffer design algorithm for production line profit maximization Original Research, *International Journal of Production Economics* 122 (2), 725-740

Smith J.M. and Daskalaki S. (1988) Buffer space-allocation in automated assembly lines. *Operations Research* **36**, 343-358.

Tan B. and Gershwin S.B. (2009) Analysis of a general Markovian two-stage continuous-flow production system with a finite buffer, *International Journal of Production Economics* **120** (2), 327-339.

Terracol C. and David R. (1987). An aggregation method for performance valuation of transfer lines with unreliable machines and finite buffers, *Proceedings of the IEEE International Conference on Robotics and Automation*, 1333–1338.