

Quantum Optimized Crossover for the Travelling Salesman and Minimization of Makespan with Setup Times

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Abstract

This paper proposes quantum-accelerated optimal recombination operators (optimized crossovers) to be used in genetic algorithms for the traveling salesman problem (TSP) and makespan minimization on a single machine with setup times. For the symmetric TSP with adjacency-based encoding, the Optimal Recombination Problem (ORP) reduces to a TSP on a graph with maximum vertex degree 4 and a set of forced edges. Applying the quantum speedup result of Moylett, Linden, and Montanaro (2017), we obtain a bounded-error quantum crossover algorithm. In the case of asymmetric TSP (ATSP) with adjacency-based encoding, the ORP reduces to the ATSP on cubic graphs, so a modification of the quantum algorithm from Moylett, Linden, and Montanaro (2017) yields a bounded-error optimized crossover. Both quantum crossovers for the TSP and ATSP have time complexity bounds, smaller by exponential factors (in the number of vertices), compared to previously known optimized crossovers for these problems. For makespan minimization on a single machine with position-based encoding, the ORP reduces to a quadratic unconstrained binary optimization (QUBO) formulation, enabling solution by quantum annealers. The QUBO has at most $\lfloor k/2 \rfloor$ binary variables and, for almost all problem instances, only $O(\log k)$ binary variables, where k is the number of jobs.

CCS Concepts

• **Theory of computation** → **Theory of randomized search heuristics**; **Quantum computation theory**.

Keywords

Genetic algorithm; optimal recombination; quantum speedup; traveling salesman problem; makespan minimization; QUBO; quantum annealing.

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1 Introduction

The performance of genetic algorithms (GAs) depends critically on the crossover operator. The *Optimal Recombination Problem* (ORP) formalizes finding the best possible offspring given two feasible parent solutions: the offspring must include every feature shared by both parents and exclude features absent in both [1, 18]. For some well-known combinatorial optimization problems the ORP is polynomially solvable, but for many other problems the ORP itself is NP-hard [10]. Computational experiments indicate that the GAs with optimized crossover, which solves the ORP exactly [9] or partition crossover operators [2] which solve it approximately, can be significantly faster than the GAs with standard randomized crossover operators.

Recent advances in quantum computing provide new tools for structured NP-hard problems. Quantum walk-based algorithms yield provable speedups for known backtracking algorithms [3, 13, 14], while quantum annealers [11] can minimize Quadratic Unconstrained Binary Optimization (QUBO) objectives arising naturally from some mathematical programming models. In this paper we apply these two approaches to design quantum-accelerated crossover operators. The main results are the following.

- (1) **TSP under adjacency-based representation.** The ORP reduces to the TSP with vertex degree at most 4 and forced edges. Application of the quantum algorithm of Moylett et al. [14] yields a bounded-error algorithm for this ORP with time complexity 1.301^n up to a polynomial factor in the problem input size.
- (2) **ATSP under adjacency-based representation.** The ORP reduces to the ATSP with vertex degree at most 3 and forced edges. Modification of a quantum algorithm from Moylett et al. [14] yields a bounded-error algorithm for this ORP with time complexity 1.233^n up to a polynomial factor in the problem input size.
- (3) **Makespan minimization with position-based representation.** The ORP reduces to a compact QUBO, amenable to quantum annealing.



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Section 2 covers the Adjacency-Based approach, Section 3 addresses Optimal Recombination with Position-Based Representation, and Section 4 concludes.

2 Optimal Recombination with Adjacency-Based Representation

2.1 ORP for the TSP

Consider a symmetric complete graph $G = (V, E)$ with $|V| = n$ vertices and edge weights $c_{ij} \geq 0$. In the *adjacency-based* (permutation matrix) encoding, each Hamiltonian cycle is uniquely represented by its edge set. Given two parent tours with edge sets E_1 and E_2 , the ORP seeks a shortest Hamiltonian cycle using only edges from $E_1 \cup E_2$ while containing all edges in $F = E_1 \cap E_2$:

$$\begin{aligned} \min \quad & \sum_{e \in U} c_e x_e \\ \text{s.t.} \quad & x \text{ defines a Hamiltonian cycle on } V, \\ & x_e = 1 \quad \forall e \in F, \quad x_e \in \{0, 1\} \quad \forall e \in U \setminus F, \end{aligned}$$

where $U = E_1 \cup E_2$. This problem is strongly NP-hard [10].

2.2 Reduction to a Degree-4 TSP with Forced Edges

In a TSP on graph (digraph) with bounded vertex degree, arbitrary positive edge (arc) weights and a given set of *forced* edges (arcs), it is required to find a shortest Hamiltonian cycle (circuit) in the given graph (digraph) that passes all forced edges (arcs).

We apply the construction from [7] (Subsection 3.2, Symmetric Case). Construct the auxiliary graph $G' = (V', E')$ as follows.

- (1) Remove all edges of $E \setminus (E_1 \cup E_2)$.
- (2) Contract every maximal path P_{uv} whose edges all lie in $F = E_1 \cap E_2$ into a single *forced edge* $\{u, v\}$ of the same total length.

Let F' denote the resulting set of forced edges in G' . Since each parent tour contributes exactly two edges incident to each vertex, every vertex of G' has degree at most 4, and $|V'| \leq n$.

An optimal solution to the TSP on G' with forced edge set F' , after expanding each forced edge back into the corresponding path, yields an optimal solution to the ORP on G .

2.3 Applying the Quantum Algorithm

Let us use the notation $O^*(c^n)$ instead of $O(n^a c^n)$ for any $a > 0$, $c > 1$, which allows to omit polynomial factors in n . Let us say that an algorithm for a TSP on graph *fails*, if either it does not find a Hamiltonian cycle when one exists or returns a non-optimal solution. Following [14], we call an algorithm that fails with a probability at most $1/3$ a *bounded-error algorithm*.

Applying the quantum speedup method from [13] to the backtracking algorithm of Xiao and Nagamochi [17], Moylett, Linden, and Montanaro [14] proved the following.

THEOREM 1 (MOYLETT ET AL., THEOREM 5). *There is a bounded-error quantum algorithm that solves the TSP on any n -vertex graph of maximum degree 4 with integer edge costs bounded by L in time $O^*(1.301^n \log L \log \log L \log(1/\delta))$.*

In fact it is easy to note in the proof of Theorem 5 in [14] that the claim of this theorem is valid for any constant failure probability $\Delta \in (0, 1)$ (not only $\Delta = 1/3$ as the above definition of a bounded-error algorithm implies). Besides that, this theorem also holds for the TSP with forced edges because Theorem 5 in [14] is based on the recursive algorithm which receives a TSP instance with a set of forced edges in each recursive call. Application of

Theorem 1 with these extensions and the set of fixed edges F' gives:

THEOREM 2. *Let the edge costs in n -vertex complete graph G be integers from $[0, L]$, then the ORP for the symmetric TSP with adjacency-based representation is solvable by a quantum algorithm in time*

$$O^*(1.301^n \log L \log \log L)$$

with failure probability at most Δ for any constant $\Delta \in (0, 1)$.

Theorem 3 is an improvement in terms of the running time over the optimized crossover operator [7], which solves this ORP in $O(1.89^n)$ time via the bounded error algorithm from [6]. The probabilistic performance guarantees do not present a significant draw back here because the GA is a randomized algorithm itself, moreover, the crossover operators may be used in the GAs with some given *crossover probability*, which is less than 1.

2.4 ORP for the Asymmetric TSP and Usage of Degree-3 Digraphs with Forced Edges

A similar approach may be taken to speed up the optimized crossover for the asymmetric version of the TSP (ATSP). Suppose a complete digraph $\bar{G}$ is given. The set of vertices of $\bar{G}$ is $V = \{v_1, \dots, v_n\}$ and a set of arcs is $A = \{(v_i, v_j) : v_i, v_j \in V, i \neq j\}$. A weight (length) $c_{ij} \geq 0$ of each arc $(v_i, v_j) \in A$ is given as well. In the ATSP, it is required to find a Hamiltonian circuit of minimum length. Below, it will be convenient to use the so called *forced ATSP* problem: Given a directed graph with non-negative lengths (weights) of arcs and a specified subset of *forced* arcs, it is required to find a minimum length Hamiltonian circuit, containing all forced arcs.

The degree of a vertex in a digraph will be understood as the total number of incoming and outgoing arcs for this vertex. A cubic digraph is the one where each vertex has a degree at most 3.

Let A_1 and A_2 be the sets of arcs of the two parent tours. In the case of adjacency-based representation of solutions to the ATSP, the ORP may be considered as the forced ATSP on the digraph $G' = (V, A_1 \cup A_2)$ with the set of forced arcs $F' := A_1 \cap A_2$. Each vertex in G' has a degree at most four. Note that the forced ATSP on the digraph G' is equivalent to the forced ATSP on the following cubic digraph $G'' = (V'', A'')$ with a set of forced arcs F'' and $|V''| \leq 2n$: Substitute each vertex of degree four $v \in V$ by a pair of vertices $\bar{v}, \tilde{v}$ connected by an *artificial arc* $(\bar{v}, \tilde{v})$ of length $c(\bar{v}, \tilde{v}) = 0$. All arcs that entered v , are now incoming in $\bar{v}$, and all arcs that left v are now outgoing from $\tilde{v}$. The set F'' consists of the set of forced arcs F' and the artificial arcs. A shortest Hamiltonian circuit of G'' respecting all forced arcs gives an optimal offspring tour after re-expanding each pseudo-arc.

To enumerate all feasible tours we consider the undirected companion graph $\bar{G} = (V'', \bar{E})$, where each symmetric arc pair in A'' becomes an edge, and forced arcs become the forced edge set $\bar{F}$. The classical algorithm of Eppstein [6] enumerates all Hamiltonian cycles of $\bar{G}$ with forced edges in time $O(n \cdot 2^{d/4})$, where $d = |\bar{E}| - |\bar{F}| \leq 2n$ is the number of arcs present in exactly one parent, giving a worst-case complexity of $O(n \cdot 2^{n/2})$. In [9] this algorithm was slightly improved to $O(2^{n/2})$, which is in $O(1.42^n)$.

2.5 Applying the Quantum Algorithm

We replace the enumeration, carried out by the algorithm of Eppstein [6], with a slightly modified quantum algorithm of Moylett, Linden and Montanaro proposed in the proof of Theorem 4 in [14] for solving the symmetric TSP on cubic graphs. We use the same quantum backtracking algorithm as in Theorem 4 [14], this algorithm is applied to graph $\bar{G}$ without any changes, but whenever it finds a Hamiltonian cycle in $\bar{G}$ (let us denote the new cycle by Q), we can check if it is possible to pass a circuit in the digraph G'' through the arcs corresponding to edges of Q in each of the two directions. Whenever this is possible, we compute the length of the circuit. Let L_T'' denote the length of the shortest circuit corresponding to Q , if such circuit exists. Bisection procedure, that calls the quantum backtracking algorithm, works analogously to that in Theorem 4 [14], but instead of the total cost of Hamiltonian cycle L_T we use the length of the circuit L_T'' , if it is defined. The circuit validation procedure for each Hamiltonian cycle may be performed in constant time if some additional information is sent to the children nodes in the backtracking tree [9]. For this modified algorithm, the failure probability bound with any constant $\Delta \in (0, 1)$ follows by the same reasoning as the failure probability bound in Theorem 4 [14]. So we have

COROLLARY 1. *There is a bounded-error quantum algorithm that solves the ATSP on any n -vertex graph of maximum degree 3 with integer edge costs bounded by L in time $O^*(1.110^n \log L \log \log L \log(1/\delta))$.*

Application of the algorithm from Corollary 1 to $\bar{G}$ with $|V''| \leq 2n$ vertices gives the following theorem.

THEOREM 3. *Let the arc costs in n -vertex complete digraph G be integers from $[0, L]$, then the ORP for the ATSP with adjacency-based representation is solvable by a quantum algorithm in time*

$$O^*(1.233^n \log(L) \log \log(L))$$

with failure probability at most Δ for any constant $\Delta \in (0, 1)$.

This theorem yields an exponential in n quantum speedup factor over the previous bound $O(1.42^n)$ [9].

3 Optimal Recombination with Position-Based Representation

3.1 ORP for Minimization of Makespan with Setup Times

The *Makespan Minimization* problem on single machine $1|s_{vu}|C_{\max}$ seeks a permutation $\pi = (\pi_1, \dots, \pi_k)$ of k jobs minimizing the total setup time $s(\pi) = \sum_{i=1}^{k-1} s_{\pi_i, \pi_{i+1}}$, where $s_{vu} \geq 0$ is the setup time from job v to job u , $s_{vu} \neq s_{uv}$ in the general case. This is equivalent to the Shortest Hamiltonian Path Problem on a complete digraph [7].

In the *position-based* encoding, component π_i denotes the i -th job in permutation π , $i = 1, \dots, k$. Given parents π^1, π^2 , the ORP seeks for permutation π^* with $\pi_i^* \in \{\pi_i^1, \pi_i^2\}$ for all $i = 1, \dots, k$ minimizing the total setup time. This ORP is NP-hard and can be reduced to the ATSP with Vertex Requisitions [7, 8].

3.2 QUBO Formulation and Quantum Optimized Crossover

Following [8], define the bipartite graph $\bar{G} = (X_k, X, \bar{U})$, where $X_k = \{1, \dots, k\}$ is the set of positions, X is the set of jobs, an edge $\{i, x\}$ exists iff $x \in \{\pi_i^1, \pi_i^2\}$. Edges of graph $\bar{G}$ are split into *special edges* (positions where $\pi_i^1 = \pi_i^2$) and $q = q(\bar{G})$ even cycles, $q \leq \lfloor k/2 \rfloor$. Each cycle i has two maximal matchings, $w^{0,i}$ and $w^{1,i}$. As a result we have 2^q perfect matchings in graph $\bar{G}$, and there is a one-to-one correspondence between the set of perfect matchings of $\bar{G}$ and the set of feasible solutions to the ORP.

A pair of vertices $(l, l+1)$ in the left-hand part X_k of $\bar{G}$, such that one of the vertices belongs to a cycle i and the other one belongs to some other cycle i' (or a special edge) will be called a *contact* between cycle i and cycle i' (or between cycle i and a special edge). A contact inside a cycle will mean a pair of vertices in the left-hand part of a cycle, if their indices differ exactly by one. If a contact $(l, l+1)$ is present inside a cycle, then each of the two maximal matchings in this cycle determines the l -th setup time of a schedule. Each combination of the maximal matchings in cycles with contacts determines the l -th setup time of a schedule for any contact $(l, l+1)$ between the cycles. Similarly we can identify setup times for contacts between cycles and special edges. Summing up such setup times we precompute (in $O(k + q^2)$ time [8]):

- P_i^m : total setup time from matching $w^{m,i}$ at contacts within cycle i and to special edges, $m \in \{0, 1\}$, $i \in \{1, \dots, q\}$;
- $P_{ii'}^{(m,m')}$: total setup time from choosing $w^{m,i}$ in cycle i and $w^{m',i'}$ in cycle i' at their contacts, $m, m' \in \{0, 1\}$, $i \neq i'$, $i, i' \in \{1, \dots, q\}$.

Introduce binary variables $d_i \in \{0, 1\}$ selecting the matching in cycle i , $i = 1, \dots, q$. The ORP may be formulated as follows:

$$\begin{aligned} \min_{d \in \{0,1\}^q} & \sum_{i=1}^q [P_i^0(1-d_i) + P_i^1 d_i] + \sum_{i=1}^{q-1} \sum_{i'=i+1}^q P_{ii'}^{(1,0)} d_i (1-d_{i'}) \\ & + \sum_{i=1}^{q-1} \sum_{i'=i+1}^q P_{ii'}^{(1,1)} d_i d_{i'} + \sum_{i=1}^{q-1} \sum_{i'=i+1}^q P_{ii'}^{(0,1)} (1-d_i) d_{i'} \\ & + \sum_{i=1}^{q-1} \sum_{i'=i+1}^q P_{ii'}^{(0,0)} (1-d_i) (1-d_{i'}). \quad (1) \end{aligned}$$

Since (1) is quadratic in binary variables with no additional constraints, it is a *Quadratic Unconstrained Binary Optimization (QUBO)* problem of size q .

The QUBO (1) is directly solvable by quantum annealers (e.g., D-Wave [5]). Heuristic nature of the methods is applicable for evolutionary algorithms, where an optimized crossover operator is applied at each iteration with some probability. Note that the obtained QUBO has $q \leq \lfloor k/2 \rfloor$ binary variables. Moreover, the results of Serdyukov [16] and Ereemeev-Kovalenko [8] state, that for almost all pairs of parent permutations $q \leq 1.1 \ln k$, so the QUBO has $O(\log k)$ variables. Also remark that the QUBO has no equality or inequality constraints, so no penalty terms are needed and the formulation is native to quantum annealers.

So we provide a quantum crossover operator for the position-based ORP for the makespan minimization on the single machine with setup times $1|s_{vu}|C_{\max}$:

- (1) Build $\bar{G}$; find special edges and cycles in $O(k)$ time.
- (2) Compute QUBO matrix in $O(k + q^2)$ time.
- (3) Submit the QUBO to a quantum annealer; obtain binary vector d^* .
- (4) Recover the optimal permutation π^* from d^* in $O(k)$ time.

A similar QUBO and solving method can be constructed for the ORP with position-based representation for the asymmetric and symmetric TSP. Here we encode solutions by permutations of vertices in positions of tours, fixing the first vertex in the first position. And placing the vertex with the minimum number among two adjacent to the first vertex in the last position in the symmetric case. Such encoding allows us to exclude different permutations for the same solution.

4 Conclusion

We have introduced two quantum-accelerated approaches to optimal recombination in genetic algorithms. For the TSP with adjacency-based representation, applying the quantum walk algorithm of [14] yields bounded-error optimized crossover operators, improving on previously known time-complexity bounds. For makespan minimization on a single machine with position-based representation, the natural QUBO reduction [8] allows quantum annealing with at most $\lfloor k/2 \rfloor$ binary variables—typically only $O(\log k)$.

These results show that quantum computing can provide meaningful speedups for the inner loop of GAs on NP-hard permutation problems. Note that the ORP instances produced during the run of a GA may be treated as subproblems of the original problem, which have sufficiently small size to be solvable sufficiently fast. A somewhat similar approach is used in the industrial hybrid solvers such as D-Wave, where classical metaheuristics (tabu search or simulated annealing) are combined with intelligent allocation of the quantum processing units to parts of the problem [12, 15].

Experimental evaluation in [4] have shown that quantum backtracking algorithms could substantially outperform leading classical competitors in terms of runtime for graph colouring and Boolean satisfiability problem instances. However the space usage of the algorithms is extremely high. Future directions include experimental validation of the proposed crossover operators on quantum hardware, hybrid classical–quantum schemes of the genetic algorithms for the considered TSP and scheduling problems, and extension to multi-machine scheduling and vehicle routing problems.

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References

- [1] E. Balas, W. Niehaus. Optimized crossover-based genetic algorithms for the maximum cardinality and maximum weight clique problems. *Journal of Heuristics* 4(2), 107–122, 1998.
- [2] O. Q. de Carvalho and D. Whitley. Dramatically Faster Partition Crossover for the Traveling Salesman Problem. In *Proceedings of the Genetic and Evolutionary Computation Conference (GECCO '25)*. Association for Computing Machinery, New York, NY, USA, 818–826, 2025.
- [3] C. Cade, A. Montanaro, and A. Belovs. Time and space efficient quantum algorithms for detecting cycles and testing bipartiteness. *Quantum Inf. Comput.*, 18(1,2):18–50, 2018.
- [4] Campbell E., Khurana A., and Montanaro A. Applying quantum algorithms to constraint satisfaction problems *Quantum*, 3: 167–197, 2019.
- [5] D-Wave Systems. D-Wave Advantage System Datasheet, 2022. <https://www.dwavesys.com>.
- [6] D. Eppstein. The travelling salesman problem for cubic graphs. *J. Graph Algorithms and Applications*, 11(1):61–81, 2007.
- [7] A.V. Eremeev and J.V. Kovalenko. Optimal recombination in genetic algorithms for combinatorial optimization problems: Part II. *Yugoslav J. Operations Research*, 24(2):165–186, 2014.
- [8] A.V. Eremeev and Yu.V. Kovalenko. On solving travelling salesman problem with vertex requisitions. *Yugoslav J. Operations Research*, 27 (4): 413–424, 2017.
- [9] A.V. Eremeev, Y.V. Kovalenko. A memetic algorithm with optimal recombination for the asymmetric travelling salesman problem. *Memetic Computing*. 12: 23–36, 2020.
- [10] A.V. Eremeev. On complexity of optimal recombination for the travelling salesman problem. In *Proc. EvoCOP 2011*, LNCS 6622, pp. 215–225. Springer, 2011.
- [11] E. Farhi, J. Goldstone, and S. Gutmann. A quantum approximate optimization algorithm. *arXiv:1411.4028*, 2014.
- [12] K. Kurowski et al. Hybrid Quantum Annealing Heuristic Method for Solving Job Shop Scheduling Problem. In: Krzhizhanovskaya, V., et al. *Computational Science – ICCS 2020*. Lecture Notes in Computer Science, vol 12142. Springer, Cham, pp. 502–515. 2020
- [13] A. Montanaro. Quantum-Walk Speedup of Backtracking Algorithms. *Theory of Computing*, 14(15):1–24, 2018.
- [14] D.J. Moylett, N. Linden, and A. Montanaro. Quantum speedup of the travelling salesman problem for bounded-degree graphs. *Physical Review A*, 95(3):032323, 2017.
- [15] A. Narimani, S.S.C. Rezaei, and A. Zaribafyan. Combinatorial Optimization by Decomposition on Hybrid CPU–non–CPU Solver Architectures. *arXiv:1708.03439*.
- [16] A.I. Serdyukov. On travelling salesman problem with prohibitions. *Upravlaemye systemi*, 17:80–86, 1978.
- [17] M. Xiao and H. Nagamochi. An improved exact algorithm for TSP in graphs of maximum degree 4. *Theory of Computing Systems*, 58(2):241–272, 2016.
- [18] M. Yagiura, T. Ibaraki. The use of dynamic programming in genetic algorithms for permutation problems. *Eur. Jour. Oper. Res.* 92, 387–401, 1996.