Forbidden subgraphs characterization of clustering graphs. Hereditary systems and matroids.

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Hereditary system

U is nonempty finite set, $\mathbf{A} \subseteq \mathbf{2^U}$ is nonempty indexed family of subsets with *hereditary axiom*

$$A_1 \in \mathbf{A}, A_2 \subseteq A_1 \Rightarrow A_2 \in \mathbf{A}.$$

A is *independent subsets*.

Set $\mathbf{D} = \mathbf{2^U} \setminus \mathbf{A}$ is satisfied

$$D_1 \in \mathbf{D}, D_1 \subseteq D_2 \Rightarrow D_2 \in \mathbf{D}.$$

D is *dependent subsets*.

B is *bases* or maximal independent subsets.

C is *circuities* or minimal dependent subsets.

$S = (U, \mathbf{A})$, $S = (U, \mathbf{B})$, $S = (U, \mathbf{C})$ and $S = (U, \mathbf{D})$ are equals *hereditary system*.

Matroids

$S = (U, \mathbf{A})$ is hereditary system and $W \subseteq U$. Base of W – each maximal independent subset of W.

S is a *matroid* if bases' cardinalities of each $W \subseteq U$ are equals. Bases' cardinality of U is *matroid rank*.

Theorem. S is a matroid $\Leftrightarrow$

(A) $A_1, A_2 \in \mathbf{A}, |A_2| = |A_1| + 1 \Rightarrow \exists a \in A_2 \setminus A_1 : A_1 \cup \{a\} \in \mathbf{A}$;

(B) $B_1, B_2 \in \mathbf{B}, b_1 \in B_1 \setminus B_2 \Rightarrow b_2 \in B_2 \setminus B_1 : (B_1 \setminus \{b_1\}) \cup \{b_2\} \in \mathbf{B}$;

(C) $C_1, C_2 \in \mathbf{C}, C_1 \neq C_2, c \in C_1 \cap C_2 \Rightarrow \exists C \in \mathbf{C} : C \subseteq (C_1 \cup C_2) \setminus \{c\}$;

(D) $D_1, D_2 \in \mathbf{D}, D_1 \cap D_2 \notin \mathbf{D}, d \in D_1 \cap D_2 \Rightarrow (D_1 \cup D_2) \setminus \{d\} \in \mathbf{D}$.

Matroids and closure operator

U is nonempty set. **cl**: $\mathbf{2^U} \to \mathbf{2^U}$ is *closure operator* if

(cl1) $X \subseteq \mathbf{cl}(X)$,

(cl2) $X \subseteq Y \Rightarrow \mathbf{cl}(X) \subseteq \mathbf{cl}(Y)$,

(cl3) $\mathbf{cl}(\mathbf{cl}(X)) = \mathbf{cl}(X)$

U and **cl**: $\mathbf{2^U} \to \mathbf{2^U}$ are *combinatorial geometry* then for each $u, v \in U$ and $X \subseteq U$:

(cl4) $v \notin \mathbf{cl}(X)$, $v \in \mathbf{cl}(X \cup \{u\}) \Rightarrow u \in \mathbf{cl}(X \cup \{v\})$,

(cl5) $\mathbf{cl}(\varnothing) = \varnothing$, $\mathbf{cl}(\{u\}) = \{u\}$ for each $u \in U$.

Matroids and closure operator

Theorem.
1) Let $M = (U, \mathbf{A})$ be a matroid. Then operator **cl**: $\mathbf{2^U} \to \mathbf{2^U}$ with

$$\mathbf{cl}(X) = X \cup \{v \in U \mid \exists A \subseteq X : A \in \mathbf{A}, A \cup \{v\} \notin \mathbf{A}\} \quad (1)$$

satisfies (cl1) – (cl4) and

$$\mathbf{A} = \{A \subseteq U \mid a \notin \mathbf{cl}(A \setminus \{a\}) \text{ for each } a \in A\} \quad (2).$$

2) Let $M = (U, \mathbf{cl})$ defined with (1) be a *combinatorial pregeometry*. Then $\mathbf{A}$ defined with (2) is indexed family of independent subsets of matroid.

Matroids and rank function

$M = (U, \mathbf{A})$ is matroid. $\mathbf{r}: \mathbf{2}^{\mathbf{U}} \to \mathbb{Z}_+$ is *rank function* that for each $X \in U$ matches $\mathbf{r}(X)$ – cardinality of arbitrary base $B \subseteq X$. Wherein

$$\mathbf{A} = \{A \subseteq U \mid \mathbf{r}(A) = |A|\} \ (1).$$

Theorem.

1) $M = (U, \mathbf{A})$ is matroid. Then rank function $\mathbf{r}: \mathbf{2}^{\mathbf{U}} \to \mathbb{Z}_+$ defined with

$$\mathbf{r}(X) = \max\{|A| : A \subseteq X, A \in \mathbf{A}\} \ (2).$$

for each $X, Y \subseteq U$ satisfies:

(r1) $\mathbf{r}(X) \leq |X|$,

(r2) $X \subseteq Y \Rightarrow \mathbf{r}(X) \leq \mathbf{r}(Y)$,

(r3) $\mathbf{r}(X \cup Y) + \mathbf{r}(X \cap Y) \leq \mathbf{r}(X) + \mathbf{r}(Y)$.

2) $\mathbf{r}$ is rank function satisfied (r1)-(r3). Then $\mathbf{A}$ defined with (1) is indexed family of independent subsets of matroid.

Cluster graph

Graph $G = (V, E)$ is cluster graph if each of its component is clique (*cluster*).

For each graph $G = (V, E)$ lets define 3 classes of cluster graphs.

Class 1. Cluster graphs C with no more than $|V|$ clusters.

Class 2. Cluster graphs $C_{\leq k}$ with no more than $2 \leq k \leq |V|$ clusters.

Class 3. Cluster graphs C_k with $2 \leq k \leq |V|$ clusters.

Hereditary system of graph

$S_G = (V, \mathbf{A}_G)$ is hereditary system of a graph $G = (V, E)$ if $\mathbf{A}_G$ is a set of *independent sets.* Circuities of S_G is edges of the G.

Theorem. Hereditary system $S_G = (V, \mathbf{A}_G)$ of a graph $G = (V, E)$ is a matroid $\Leftrightarrow$ G is a cluster graph.

Forbidden subgraphs characterization of **Class 1**

$C = (V, E)$ is a cluster graph of **Class 1**. Each $u,v,w \in E$ cannot form *induced subgraph* P_2.

Forbidden subgraphs characterization of **Class 1**

Theorem. $C = (V, E)$ is a cluster graph of **Class 1** $\Leftrightarrow$ C is P_2 free graph.

Proof. => Obviously.

<= Lets proof that $(u,v) \notin E$ => u and v belongs to different components of C.

If $\exists w \in V$: $(u,w) \in E$ and $(v,w) \in E$ => u,v,w is P_2. Contradiction.

If $\exists$ $(u = w_1, w_2, \ldots, w_n = v)$ is a shortest simple chain => $\exists\, i, i + 1, i + 2$ such that w_i, w_{i+1}, w_{i+2} is P_2 (or else $(u = w_1, \ldots, w_i, w_{i+2}, \ldots, w_n = v)$ is shorter. Contradiction.

Boolean programming

$x_{ij} = 0$ if i and j belongs to the same cluster and $x_{ij} = 1$ otherwise.

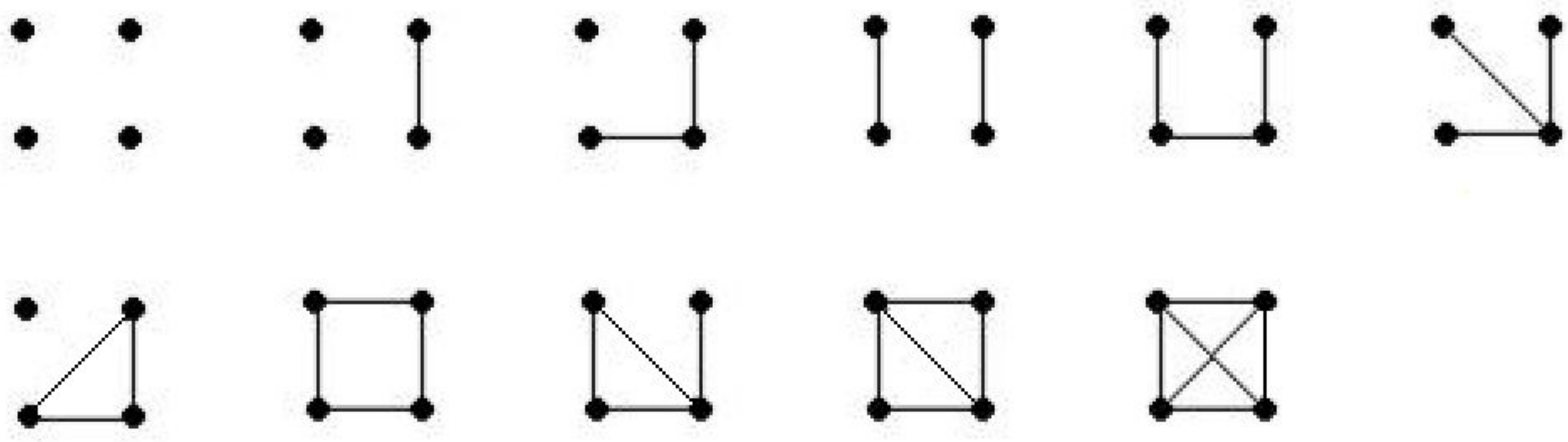
$$\sum_{ij \in E} x_{ij} + \sum_{ij \notin E} (1 - x_{ij}) \rightarrow \min$$

$$x_{ir} \leq x_{ij} + x_{jr},\ i, j, r \in \{1, \ldots, n\}$$

$$x_{ij} \in \{0,1\},\ i, j \in \{1, \ldots, n\}$$

Forbidden subgraphs characterization of **Class 2**

$C = (V, E)$ is a cluster graph of **Class 2**. Each $u,v,w \in E$ cannot form induced subgraph P_2. Each u_1, u_2, …, u_k cannot form induced subgraph O_{k+1}.



Forbidden subgraphs characterization of **Class 2**

Theorem. $C = (V, E)$ is a cluster graph of **Class 2** $\Leftrightarrow$ C is P_2 free and O_{k+1} free graph.

$$\sum_{ij\in E} x_{ij} + \sum_{ij\notin E} (1 - x_{ij}) \to \min$$

$$x_{ir} \le x_{ij} + x_{jr},\ i, j, r \in \{1, \ldots, n\}$$

$$x_{i_1 i_2} + \ldots + x_{i_{k-1} i_k} \le \frac{(k+2)(k-1)}{2},\ i_1, \ldots, i_k \in \{1, \ldots, n\}$$

$$x_{ij} \in \{0,1\},\ i, j \in \{1, \ldots, n\}$$

Open questions

1. Forbidden subgraphs characterization of **Class 3**.
2. Rank function for hereditary systems.

Thank you for your attention!