A method of lines for elliptic problem with a boundary layers along a strip*

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Аннотация

Elliptic equation with parabolic boundary layers along a half-strip is considered. Method of lines, taking into account boundary layers, is used to transform a problem to a system of ordinary differential equations on half-infinite interval. Method of extraction of a set of solutions, satisfying the limit conditions at the infinity, is used to transform a problem to a finite interval. Asymptotic series are used to solve auxiliary Couchy problems with conditions at an infinity.

1 Introduction

We consider a singular perturbed elliptic equation on a half-strip. Solution of a problem has boundary layers along a strip. If we shall construct a difference scheme for this problem, we'll have a scheme with an infinite number of mesh points, that is bad for computer calculations. The purpose of the article is to transform a problem to a problem for bounded domain.

Introduce some designations. Let C and C_i be constants, undepending on parameter ε and mesh steps. For bounded functions $p(x),\ q(x,y)$ $||p|| = \max\limits_x |p(x)|,\ \ ||q|| = \max\limits_{x,y} |q(x,y)|$.

Let us formulate the problem under consideration. We consider the Dirichlet problem for the following singularly perturbed elliptic equation:

$$\varepsilon\frac{\partial^2 u}{\partial x^2} + \varepsilon\frac{\partial^2 u}{\partial y^2} - a(x,y)\frac{\partial u}{\partial x} - b(x,y)u = f(x,y), \tag{1}$$

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$$u(x,y) = \phi_i,\ (x,y) \in l_i\ i = 1,2,3,\quad \lim_{x\to\infty} u(x,y) = 0 \tag{2}$$

for half-infinite strip $D = \{0 \le x < \infty,\ 0 \le y \le 1\}$, where l_1, l_2, l_3 - linear parts of the boundary:

$$l_1 = \{y = 0,\ 0 \le x < \infty\},\ l_2 = \{x = 0,\ 0 \le y \le 1\},\ l_3 = \{y = 1,\ 0 \le x < \infty\}.$$

Functions $a, b, f,\ \phi_i$ are sufficiently smooth on D,

$$\varepsilon > 0,\ a(x,y) \ge \alpha > 0,\ b(x,y) \ge \beta > 0, \tag{3}$$

$$\lim_{x\to\infty} \phi_i(x) = 0,\ i = 1,3,\quad \lim_{x\to\infty} f(x,y) = 0,$$

$$\lim_{x\to\infty} a(x,y) = a_0(y),\quad \lim_{x\to\infty} b(x,y) = b_0(y), \tag{4}$$

$a_0(y), b_0(y)$ are continuous functions.

Get some astimates on solution $u(x,y)$ and its derivatives. Using maximum principle, we get estimate:

$$||u|| \le ||f||/\beta + \max_i ||\phi_i||. \tag{5}$$

We use known estimates of derivatives, for example, from [5]:

$$\left|\frac{\partial u}{\partial x}\right|,\quad \left|\frac{\partial^2 u}{\partial x^2}\right| \le C_1, \tag{6}$$

$$\left|\frac{\partial^j}{\partial y^j} u(x,y)\right| \le C_2 \left[1 + \varepsilon^{-j/2}\left(\exp\{-(m/\varepsilon)^{1/2} y\} + \exp\{(m/\varepsilon)^{1/2}(y-1)\}\right)\right],$$

$$\beta/2 < m < \beta,\ j = 1,2,3,4. \tag{7}$$

According to (7), the solution of the problem (1)-(2) has the boundary layers along a half-strip.

2 Method of lines

Introduce nonuniform mesh in variable y :

$$\Omega_y = \{y_j, j = 0,1,...,N,\ h_j = y_j - y_{j-1}\}.$$

We approximate equation (1) in variable y and get a system of differential - difference equations:

$$L_j\mathbf{V}(x) = \varepsilon\frac{d^2V_j}{dx^2} + \varepsilon\Lambda_{yy,j}\mathbf{V} - a(x,y_j)\frac{dV_j}{dx} - b(x,y_j)V_j = f(x,y_j),\ 0 < j < N,$$

$$V_j(0) = \phi_2(y_j),\ \lim_{x\to\infty} V_j(x) = 0,\ V_0(x) = \phi_1(x),\ V_N(x) = \phi_3(x), \quad (8)$$

where $\mathbf{V}(x) = (V_0(x), V_1(x), \ldots, V_N(x))$,

$$\Lambda_{yy,j}\mathbf{V} = \frac{h_j(V_{j+1} - V_j) - h_{j+1}(V_j - V_{j-1})}{h_j h_{j+1}(h_j + h_{j+1})/2}.$$

We take into account boundary layers along a strip and use Bakhvalov mesh in the variable y [1]:

$$y_j = \lambda(t_j),\ t_j = j/N,\ j = 0, 1, ..., N. \quad (9)$$

Define $\mathbf{U} = [u]_{\Omega_y}$.

Lemma 1. *For some constant C*

$$\max_i \max_x |U_i(x) - V_i(x)| \le \frac{C}{N^2}. \quad (10)$$

Proof. Define $\mathbf{Z} = \mathbf{V} - \mathbf{U}$. Then

$$L_j\mathbf{Z}(x) = \varepsilon\frac{\partial^2 u}{\partial y^2}(x,y_j) - \varepsilon\Lambda_{yy,j}[u]_{\Omega_y}, \quad j = 1, ..., N-1,$$

$$\mathbf{Z}(0) = \mathbf{0}, \quad \lim_{x\to\infty}\mathbf{Z}(x) = \mathbf{0}, \quad z_0(x) = 0,\ z_N(x) = 0. \quad (11)$$

Using estimates of derivatives (7), as in the case of ordinary differential equation in [1], on Bakhvalov mesh Ω_y we get

$$\max_j \max_x |L_j\mathbf{Z}(x)| \le \frac{C}{N^2}.$$

Applying maximum principle, we get estimate (10). Lemma is proved.

We can use other nonuniform mesh in variable y. If we'll use more compact Shishkin's mesh [2], we'll have instead of (10) next estimate:

$$\max_i \max_x |U_i(x) - V_i(x)| \le \frac{C}{N^2}\ln^2 N. \quad (12)$$

3 Transfer of the boundary condition from infinity

Consider a system of equations in matrix form:

$$T_\varepsilon \mathbf{V}(x) = \varepsilon \mathbf{V}''(x) - a(x)\mathbf{V}'(x) - \mathbf{M}(x)\mathbf{V}(x) = \mathbf{F}(x), \tag{13}$$

$$\mathbf{V}(0) = \mathbf{A}, \quad \lim_{x\to\infty} \mathbf{V}(x) = \mathbf{0}, \tag{14}$$

where $a(x) \geq \alpha > 0$, $\mathbf{M}(x)$ is positive definite matrix of order K. If we used method of lines, then $K = N - 1$. Vector-function $\mathbf{F}(x) \to \mathbf{0}$, $x \to \infty$,

$$(\mathbf{M}(x)\mathbf{y}, \mathbf{y}) \geq \beta(\mathbf{y}, \mathbf{y}). \tag{15}$$

Get estimate of stability for problem (13)-(14).

Lemma 2. *Next estimate has a place:*

$$||\mathbf{V}(x)||^2 \leq ||\mathbf{A}||^2 + \frac{1}{\beta^2} \max_x ||\mathbf{F}(x)||^2, \quad ||\mathbf{U}||^2 = \sum_{i=1}^{K} U_i^2. \tag{16}$$

Proof. Let $w(x) = ||\mathbf{V}(x)||^2$. Multiply equation (13) on $\mathbf{V}(x)$ and get:

$$\frac{\varepsilon}{2} w''(x) - \frac{a(x)}{2} w'(x) - \beta w(x) = (\mathbf{MV}, \mathbf{V}) - \beta(\mathbf{V}, \mathbf{V}) + \varepsilon(\mathbf{V}', \mathbf{V}') + (\mathbf{F}, \mathbf{V}).$$

We use known inequality

$$|2(\mathbf{F}, \mathbf{V})| \leq \sigma ||\mathbf{F}||^2 + \frac{1}{\sigma} ||\mathbf{V}||^2, \quad \sigma > 0.$$

Taking $\sigma = 1/\beta$, we get

$$Lw(x) = \varepsilon w''(x) - a(x)w'(x) - \beta w(x) \geq -\frac{1}{\beta} ||\mathbf{F}||^2. \tag{17}$$

Define

$$\Psi(x) = \frac{1}{\beta^2} \max_x ||\mathbf{F}(\mathbf{x})||^2 + ||\mathbf{A}||^2 - w(x).$$

Using estimate (17), we get:

$$L\Psi(x) \leq 0, \ x > 0, \quad \Psi(0) \geq 0, \ \lim_{x\to\infty} \Psi(x) \geq 0.$$

$>$From maximum principle follows $\Psi(x) \geq 0, x \geq 0$. Lemma is proved.

Transform a problem (13)-(14) to a problem for a finite interval. We use approach, developed in works of Abramov A.A., Balla K., Konyukhova N.B., for example [3]. For equations with a small parameter we used that approach, for example, in [4]. Define a set of solutions of a system (13), satisfying the limit condition at infinity (14):

$$\mathbf{V}'(x) = \gamma(x)\mathbf{V}(x) + \theta(x), \tag{18}$$

where $\gamma(x)$ is solution of matrix Riccati differential equation

$$\varepsilon\gamma' + \varepsilon\gamma^2 - a(x)\gamma - \mathbf{M}(x) = \mathbf{0}, \quad \gamma(x) \to \gamma_\infty, \ x \to \infty, \tag{19}$$

$\theta(x)$ is solution of linear problem:

$$\varepsilon\theta' - [a(x)\mathbf{I} - \varepsilon\gamma(x)]\theta = \mathbf{F}(x), \quad \theta(x) \to \mathbf{0}, \ x \to \infty, \tag{20}$$

where γ_∞ is solution of quadratic matrix equation

$$\varepsilon\gamma^2 - a_\infty\gamma - \mathbf{M}_\infty = \mathbf{0}, \quad \gamma_\infty = -2\mathbf{M}_\infty \left[a_\infty\mathbf{I} + \sqrt{a_\infty^2\mathbf{I} + 4\mathbf{M}_\infty\varepsilon}\right]^{-1}. \tag{21}$$

According to known results, problems (19),(20) have unique solution for $x > 0$. Using (18), transform problem (13)-(14) to a finite interval:

$$\varepsilon\mathbf{V}'' - a(x)\mathbf{V}' - \mathbf{M}(x)\mathbf{V} = \mathbf{F}(x),$$

$$\mathbf{V}(0) = \mathbf{A}, \quad \mathbf{V}'(S) + \mathbf{g}\mathbf{V}(S) = \theta(S), \quad \mathbf{g} = -\gamma(S). \tag{22}$$

where $(\mathbf{gy}, \mathbf{y}) \geq \delta(\mathbf{y}, \mathbf{y}), \ \delta > 0, \ \mathbf{y} \in R^K$.

Problems (13)-(14) and (22) have a same solution on interval $[0, S]$. To prove it, consider an initial value problem:

$$\mathbf{V}'(x) - \gamma(x)\mathbf{V}(x) = \theta(x), \quad \mathbf{V}(0) = \mathbf{A}. \tag{23}$$

We take into account, that problems (13)-(14) and (22) have unique solution. On other hand, solution of problem (23) satisfies to problems (13)-(14) and (22). So, (13)-(14) and (22) have a same solution.

Coefficients $\gamma(S), \theta(S)$ from problems (19),(20) can be found only approximately. We need to get estimate of stability for problem (22) to errors in $\gamma(S)$ and $\theta(S)$.

Lemma 3. *Let $\tilde{\mathbf{V}}(x)$ is solution of problem (22) with coefficients $\tilde{\gamma}(S)$ and $\tilde{\theta}(S)$. Let matrix norm is in agreement with vector norm,*

$$||\theta(S) - \tilde{\theta}(S)|| \leq \Delta, \ ||\mathbf{g} - \tilde{\mathbf{g}}|| \leq \Delta, \ (\tilde{\mathbf{g}}\mathbf{y}, \mathbf{y}) \geq \tilde{\delta}(\mathbf{y}, \mathbf{y}), \ \tilde{\delta} > 0, \ \mathbf{y} \in R^K.$$

Then for some constant C

$$||\mathbf{V}(x) - \tilde{\mathbf{V}}(x)|| \leq C\Delta\sqrt{\varepsilon}\exp(\alpha\varepsilon^{-1}(x-S)/2). \tag{24}$$

Proof. Let $\mathbf{Z}(x) = \mathbf{V}(x) - \tilde{\mathbf{V}}(x)$. Then

$$T_\varepsilon \mathbf{Z}(x) = 0, \ \ \mathbf{Z}(0) = \mathbf{0}, \ \ \mathbf{Z}'(S) + \tilde{\mathbf{g}}\mathbf{Z}(S) = \theta(S) - \tilde{\theta}(S) + (\tilde{\mathbf{g}} - \mathbf{g})\mathbf{V}(S). \tag{25}$$

We multiply (25) on $\mathbf{Z}(x)$, introduce $w(x) = ||\mathbf{Z}(x)||^2$ and get

$$w(0) = 0, \ Lw = \varepsilon w'' - aw' - \beta w \geq 0, \ Rw = w'(S) + \tilde{\theta}(S)w(S) \leq C_0\Delta^2.$$

Let

$$\Psi(x) = C\varepsilon\Delta^2\exp(\alpha\varepsilon^{-1}(x-S)) - w(x).$$

For some constant C

$$\Psi(0) \geq 0, \ L\Psi(x) \leq 0, \ 0 < x < S, \ \ R\Psi(x) \geq 0.$$

It follows from maximum principle, that $\Psi(x) \geq 0$. Lemma is proved.

We find coeffitients $\gamma(S), \theta(S)$ from problems (19),(20) in a form:

$$\gamma^m(x) = \sum_{k=0}^{m} \gamma_k \varepsilon^k, \ \ \theta^m(x) = \sum_{k=0}^{m} \theta_k \varepsilon^k.$$

Using this series in (19),(20), we get reccurent formulas on γ_k, θ_k.

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