A SECOND ORDER SCHEME FOR NONLINEAR SINGULARLY PERTURBED
TWO-POINT BOUNDARY VALUE PROBLEM

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Аннотация

An nonlinear second order ordinary singularly perturbed equation is consided.We suppose that the solution has boundary layers near the endpoints of the interval. We use the linear interpolation for the source in the equation for the every mesh interval. The concordance of the solutions in the neighbouring mesh intervals leeds to the finite-difference scheme. We prove that under some conditions this scheme has the second order of the accuracy uniformly in a small parameter.

We shall consider the following singularly perturbed boundary value problem:

$$Lu = \epsilon u'' - f(x.u) = 0, u(0) = A, u(1) = B \tag{1}$$

with the hypotheses, which will be assumed throughout the paper :

$$\epsilon > 0, f \in C^2(I, R), I = [0, 1], f'_u \geq \beta > 0. \tag{2}$$

The solution $u(x)$ has two boundary layers near the endpoints of the interval I.

In the linear case of problem (1) the question of construction of a difference scheme with the property of uniform convergence in a small parameter ϵ at first was considered by Bahvalov N.S [1]. In that paper was proved the uniform convergevce of the central difference scheme on the mesh with the points concentrated in the boundary layers. This approach, based on the concentration of mesh points in the regions where the solution changes rapidly, was developed by Liseikin V.D., Petrenko B.E.[2]. In [3] the numerical soluton of nonlinear problem (1) was found by using Richardson extrapolation on a special nonuniform discretizaton mesh.In [4] after some transformation the problem (1) was solved by a standard difference scheme on the equidistant mesh in the combination with the solution of the corresponding reduced problem.

Another approach for construction of the schemes with the property of uniform convergence in a small parameter ϵ - to take into account boundary-layer behavior of the solution in the coefficients of the difference scheme. An exact difference scheme was founded by Samarsky A.A. [5]. Familiar approach was presented by Gaevoy V.P.[6] . In the linear case the special difference scheme was constructed in [7], [8] and in some other papes [9].

In this paper for nonlinear problem (1) we'll construct the scheme of the second order accuracy uniformly in ϵ . Our method corresponds to the second from the mentioned approaches. Throughout the paper C and C_i denote positive constants independing of ϵ and mesh steps. We use the maximum norm $\|V\| = \max |V_i|$ for any vector $V = (V_1, V_2, ..., V_N)$ and

$\|V\| = \max |V(x)|, x \in I$ for any function $V(x)$.

We introduce the mesh Ω :

$$\Omega = \{x_i : x_i = x_{i-1} + h_i, x_0 = 0, x_N = 1\}, \ \Delta_i = [x_i, x_{i+1}).$$

To construct the difference scheme for the problem (1) we consider the problem:

$$\widetilde{L}V = \epsilon V'' - \widetilde{f}(x, V) = 0, V(0) = A, V(1) = B, \tag{3}$$

where for $x \in \Delta_n$

$$\widetilde{f}(x, V) = f_n + f'_{xn}(x - x_n) + f'_{Vn}(V - V_n)$$

with

$$f_n = f(x_n, V_n), \ f'_{xn} = f'_x(x_n, V_n), \ f'_{Vn} = f'_V(x_n, V_n), V_n = V(x_n).$$

For every interval Δ_n the solution of the problem (3) has a form :

$$V_n(x) = V_n^0(x) + \gamma_1^{(n)} \exp[\alpha_n(x - x_n)] + \gamma_2^{(n)} \exp[-\alpha_n(x - x_n)], \tag{4}$$

where V_n^0- some solution of the equation (3) :

$$V_n^0(x) = -\beta_n^{-2}(b_n x + d_n) + \beta_n^{-2}(b_n x_n + d_n) \cosh[\alpha_n(x - x_n)] +$$

$$+ b_n \beta_n^{-2} \alpha_n^{-1} \sinh[\alpha_n(x - x_n)],$$

where

$$\alpha_n = \beta_n \epsilon^{-0.5}, \beta_n = (f'_{Vn})^{0.5}, b_n = f'_{xn}, d_n = f_n - b_n x_n - f'_{Vn} V_n.$$

We select $\gamma_1^{(n)}, \gamma_2^{(n)}$ to satisfy conditions of continuity for $V(x)$:

$$V_n(x_n) = V_n^h, V_n(x_{n+1}) = V_{n+1}^h.$$

The condition $V \in C^1[0,1]$ is valid, if

$$V_{n-1}'(x_n) = V_n'(x_n), \ n = 1, 2, ..., N-1.$$

Using (4), we obtain the system of finite-difference equations :

$$\frac{\beta_n}{\sinh(\alpha_n h_{n+1})} V_{n+1}^h - \left[\frac{\beta_n}{\sinh(\alpha_n h_{n+1})} + \beta_{n-1} \coth(\alpha_{n-1} h_n) \right] V_n^h +$$

$$+\beta_{n-1} \coth(\alpha_{n-1} h_n) V_{n-1}^h = F_n, \ V_0^h = A, \ V_N^h = B, \ n = 1, 2, ..., N-1, \quad (5)$$

where

$$F_n = \frac{f_{n-1}}{\beta_{n-1}} \tanh \frac{\alpha_{n-1} h_n}{2} + \frac{f_n}{\beta_n} \tanh \frac{\alpha_n h_{n+1}}{2} +$$
$$+ \frac{b_n h_{n+1}}{\beta_n} \left[\frac{1}{\alpha_n h_{n+1}} - \frac{1}{\sinh(\alpha_n h_{n+1})} \right] +$$
$$+ \frac{b_{n-1} h_n}{\beta_{n-1}} \frac{\alpha_{n-1} h_n \coth(\alpha_{n-1} h_n) - 1}{\alpha_{n-1} h_n}.$$

The difference scheme (5) is a system of nonlinear algebraic equations. The solution of this system is bounded, if some conditions on the function f are assumed.

Lemma 1. Suppose that

$$f = f(u), f''(u)u \geq 0. \quad (6)$$

Then

$$\|V^h\| \leq \max\{|A|, \ |B|, \ |f(0)|/\beta\}. \quad (7)$$

Proof. Rewrite scheme (5) in a form :

$$L_n^h V^h = \frac{\beta_n}{\sinh(\alpha_n h_{n+1})} V_{n+1}^h - \left[\frac{\beta_n}{\sinh(\alpha_n h_{n+1})} + \right.$$

$$\left. + \beta_{n-1} \coth(\alpha_{n-1} h_n) + \frac{f'(\theta_n)}{\beta_n} \tanh \frac{\alpha_n h_{n+1}}{2} \right] V_n^h +$$

$$+\left[\beta_{n-1}\coth(\alpha_{n-1}h_n)-\frac{f'(\kappa_n)}{\beta_{n-1}}\tanh\frac{\alpha_{n-1}h_n}{2}\right]V^h_{n-1}=\frac{f(0)}{\beta_{n-1}}\tanh\frac{\alpha_{n-1}h_n}{2}+$$

$$+\frac{f(0)}{\beta_n}\tanh\frac{\alpha_n h_{n+1}}{2},\ V^h_0=A,\ V^h_N=B,\ n=1,2,...,N-1,$$

where $\kappa_n\in(0,V^h_{n-1})$. Using (6) we can prove that $\beta^2_{n-1}\geq f'(\kappa_n)$. It follows that maximum principle for operator L^h is valid [5], [10] : if

$$L^h_n\Psi^h\leq 0,\ \Psi^h_0\geq 0,\ \Psi^h_N\geq 0,\ n=1,2,...,N-1\ , \tag{8}$$

then $\Psi^h_n\geq 0,\ n=0,1,...,N$. We define $\Psi^h=\max\{|A|,\ |B|,|f(0)|/\beta\}\pm V^h$. For given Ψ^h inequalities (8) have a place , it follows $\Psi^h\geq 0$. Lemma is proved.

Lemma 2. Suppose that $\|V^h\|\leq C$, where V^h is the solution of the scheme (5). Then :

$$\|V\|\leq C_1,\ |V'(x)|\leq C_2\left[1+\epsilon^{-0.5}\left(\exp(-\kappa x)+\exp(\kappa(x-1))\right)\right], \tag{9}$$

where $\kappa=(\beta/\epsilon)^{0.5}$, V is the solution of the problem (3).

Proof. Write the problem (3) in a form :

$$\epsilon V''-a(x)V=g(x),V(o)=A,V(1)=B, \tag{10}$$

where for $x\in\Delta_n$ $a(x)=\beta^2_n,\ g(x)=f_n+b_n(x-x_n)-\beta^2_nV^h_n$. Taking into account that $\|V^h\|\leq C$ we get $|g(x)|\leq C_3$. Using inequality $\|V\|\leq|A|+|B|+\beta^{-1}\|g\|$, we get $\|V\|\leq C_1$. Using the condition $|g(x)|\leq C_3$ and famillar results [7],we get (9).Lemma is proved.

Let Ω be a nonuniform mesh of the interval I.We want a relatively high density of points in a boundary layers. Let M be a number of mesh points in the boundary layer. We take for $n\leq M$

$$x_n=-\kappa^{-1}\ln[1-(1-\kappa^{-1})M^{-1}n]. \tag{11}$$

For $n\geq N-M$

$$x_n=1+\kappa^{-1}\ln[1-(1-\kappa^{-1})M^{-1}(N-n)]. \tag{12}$$

We suppose, that the mesh Ω is uniform out of boundary layers, Q is a number of uniform steps, $Q=N-2M$.

Theorem 1. Let Ω satisfies to conditions (11)-(12), V^h is a bounded solution of the scheme (5), $[u]_\Omega$ is the solution of the problem (1) on mesh points. Then

$$\|V^h - [u]_\Omega\| \le C[Q^{-2} + M^{-2}]. \tag{13}$$

Proof. Let $z = u - V$. Then

$$\epsilon z'' - [f(x,u) - f(x,V)](u-V)^{-1} z = f(x,V) - \widetilde{f}(x,V).$$

For $x \in \Delta_n$ $|f(x,V) - \widetilde{f}(x,V)| \le C[h_{n+1}^2 + |V(x) - V(x_n)|^2]$. Using the estimate (9) ,we get :

$$|V(x) - V(x_n)| \le \int_{x_n}^{x} |V'(s)| ds \le C_1 h_{n+1} +$$

$$+ C_2\{\exp[-\kappa x_n] - \exp[-\kappa x_{n+1}] + \exp[\kappa(x_{n+1} - 1)] - \exp[\kappa(x_n - 1)]\}.$$

For $n < M$ we use (11) and get :

$$|V(x) - V(x_n)| \le C_3[h_{n+1} + M^{-1}].$$

For $n < M$ $h_{n+1} < M^{-1}$. It follows

$$|V(x) - V(x_n)| \le C M^{-1}.$$

The case $n > N - M$ may be considered in the same manner.

For $M \le n \le N - M$ we get $|V(x) - V(x_n)| \le C Q^{-1}$. So, we get

$$|f(x,V) - \widetilde{f}(x,V)| \le C[M^{-2} + Q^{-2}].$$

Taking into account, that $[V]_\Omega = V^h$ and $\|z\| \le \beta^{-1}\|f(x,V) - \widetilde{f}(x,V)\|$, we get (13). Theorem is proved.

Consider the case of a linear problem :

$$Lu = \epsilon u'' - c(x)u = f(x), \; u(0) = A, \; u(1) = B, \tag{14}$$

where $c(x) \ge \beta > 0, \; \epsilon > 0, \;\; f, c \in C^2[0,1]$.

We rewrite scheme (5) for the problem (14):

$$L_n^h V^h = \frac{\beta_n}{\sinh z_{n+1}} V_{n+1}^h - \left[\frac{\beta_n}{\sinh z_{n+1}} + \beta_{n-1} \coth z_n + \frac{c_n}{\beta_n} \tanh \frac{z_{n+1}}{2} + \right.$$

$$+\frac{c_n' h_{n+1}}{\beta_n}\left(\frac{1}{z_{n+1}}-\frac{1}{\sinh z_{n+1}}\right)\Bigg] V_n^h+\left[\beta_{n-1}\coth z_n-\frac{c_{n-1}}{\beta_{n-1}}\tanh\frac{z_n}{2}-\right.$$

$$\left.-\frac{c_{n-1}' h_n}{\beta_{n-1}}\frac{z_n\coth z_n-1}{z_n}\right] V_{n-1}^h=\frac{f_{n-1}}{\beta_{n-1}}\tanh\frac{z_n}{2}+\frac{f_n}{\beta_n}\tanh\frac{z_{n+1}}{2}+$$

$$+\frac{f'(x_n)h_{n+1}}{\beta_n}\left[\frac{1}{z_{n+1}}-\frac{1}{\sinh z_{n+1}}\right]+\frac{f_{n-1}' h_n}{\beta_{n-1}}\frac{z_n\coth z_n-1}{z_n}, \qquad (15)$$

where

$$c_n=c(x_n), c_n'=c'(x_n), \beta_n^2=c_n, \alpha_n^2=\beta_n^2/\epsilon, z_n=\alpha_{n-1}h_n, f_n=f(x_n),$$

$$f_n'=f'(x_n).$$

Theorem 2. Let for any step h_n

$$h_n\le h,\ h^2\|C''\|\le\beta. \qquad (16)$$

Then the next estimate for the error of the scheme (15) is valid:

$$\|V^h-[u]_\Omega\|\le Ch^2.$$

Proof. For the problem (14) the equation (3) has a form :

$$\epsilon V''-\widetilde{c}(x)V=\widetilde{f}(x),\ V(0)=A,\ V(1)=B, \qquad (17)$$

where for $x\in\Delta_n$

$$\widetilde{c}(x)=c_n+c_n'(x-x_n),\ \widetilde{f}(x)=f_n+f_n'(x-x_n).$$

Prove estimate $\|V\|\le C$. Taking into account (16), we have :

$$\widetilde{c}(x)=c(x)-c''(\theta)(x-x_n)^2/2\ge\beta/2.$$

It imply $|V(x)|\le 2\beta^{-1}\|\widetilde{f}\|+\max\{|A|,|B|\}$. So, $\|V(x)\|\le C$. Now we estimate $z=u-V$. We have :

$$Lz=f(x)-\widetilde{f}(x)+[c(x)-\widetilde{c}(x)]V.$$

Hence, $|Lz|\le Ch^2$. Taking into account, that $\|z\|\le\beta^{-1}\|Lz\|$, we get $\|u-V\|\le Ch^2$. Using that $[V]_\Omega$ is an exact solution of scheme (15) , we complete the proof of the theorem 2.

Now we'll consider the results of the numerical experiments.

Write a boundary value problem in a case of linear equation [4]:

$$\epsilon^2 u'' - u = \cos^2 \pi x + 2(\pi\epsilon)^2 \cos(2\pi x),\ \ u(0) = 0,\ \ u(1) = 0. \tag{18}$$

The solution of (18) has a form :

$$u(x) = [\exp(-x/\epsilon) + \exp(-(1-x)/\epsilon)] \,/\, [1 + \exp(-1/\epsilon)] - \cos^2(\pi x).$$

According to the theorem 2 the scheme (15) has a property of the second order accuracy for any mesh. Let Ω be uniform mesh with the step h. We present a norm of the error $[u]_\Omega - V^h$ depending on ϵ and h in the table 1. The table 1 confirms the theorem 2. In the table 2 is presented the norm of the error for the usiall second order difference scheme. Numerical results confirm the advantage of the scheme (15).

Now we'll consider the case of nonlinear boundary-value problem :

$$\epsilon u'' - u^3 - u = f(x),\ \ u(0) = 1,\ \ u(1) = 1, \tag{19}$$

where function $f(x)$ corresponds to solution :

$$u(x) = \left[\exp(-\epsilon^{-0.5} x) + \exp(\epsilon^{-0.5}(x-1))\right] \,/\, \left[1 + \exp(-\epsilon^{-0.5})\right] + \sin(\pi x).$$

The difference scheme (5) is a system of nonlinear algebraic equations. We use modified Pikar method to find its solution [11] :

$$\frac{\beta_n^{(k)}}{\sinh(\alpha_n^{(k)} h_{n+1})} V_{n+1}^{(k+1)} - \left[\frac{\beta_n^{(k)}}{\sinh(\alpha_n^{(k)} h_{n+1})} + \beta_{n-1}^{(k)} \coth(\alpha_{n-1}^{(k)} h_n)\right] V_n^{(k+1)} +$$

$$+\beta_{n-1}^{(k)} \coth(\alpha_{n-1}^{(k)} h_n) V_{n-1}^{(k+1)} - G V_n^{(k+1)} = F_n^{(k)} - G V_n^{(k)},$$

$$V_0^{(k+1)} = A,\ V_N^{(k+1)} = B,\ n = 1, 2, ..., N-1, \tag{20}$$

As initial iteration for solution we chose $V_n^{(0)} = 1$ for any n . The numerical experiments indicated the convergence of iterative method for $G > 1$. For calculations we chose $G = 5$. We break iterative process if maximal error between solutions of neighbouring iterations not more 10^{-5}. On every iterative step we used Gaussian elimination method [5].

Table 3 contains a norm of the error for scheme (5) in the case of problem (19) depending on ϵ and M, where Њ - number of mesh steps in each boundary layer. A number of uniform steps out of baundary layers is the same as in layers ($Q = M$). In the layers mesh points were chosen according to (11),(12). The results given in Table 3 support estimate (13).

Table 4 contains a norm of the error for scheme (5) in the case of the uniform mesh. This calculations indicate that in the case of nonlinear problem and scheme (5) there is necessity to concentrate mesh points in the boundary layers.

Table 1.

ϵ	h			
	0.1	0.05	0.01	0.005
1.0	5.9E-2	1.5E-2	6.2E-4	1.5E-4
0.1	3.2E-2	8.2E-3	3.3E-4	8.3E-5
0.01	4.0E-2	9.2E-3	3.3E-4	8.2E-5
0.005	4.3E-2	1.0E-2	3.3E-4	8.3E-5

Table 2.

ϵ	h			
	0.1	0.05	0.01	0.005
1.0	3.1E-2	7.6E-3	3.0E-4	7.5E-5
0.1	1.2E-2	3.2E-3	1.3E-4	3.3E-5
0.01	9.7E-3	3.0E-2	1.4E-2	3.7E-3
0.005	2.5E-3	9.8E-3	3.6E-2	1.4E-2

Table 3.

ϵ	M			
	3	10	30	100
0.1	0.16E-1	0.17E-2	0.36E-2	0.60E-3
0.1E-1	0.49E-1	0.42E-2	0.29E-2	0.48E-3
0.1E-2	0.18	0.12E-1	0.33E-2	0.56E-3
0.1E-3	0.29	0.19E-1	0.35E-2	0.60E-3
0.1E-4	0.32	0.23E-1	0.36E-2	0.61E-3

Table 4.

ϵ	h		
	0.1	0.02	0.01
0.1	0.11E-1	0.33E-3	0.12E-3
0.01	0.71E-1	0.13E-2	0.24E-3
0.001	0.91	0.24E-1	0.38E-2
0.0001	3.3	0.38	0.81E-1

Список литературы

[1] Bahvalov N.S. About the optimization of methods for solution of boundary value problems with boundary layer// Jornal of comput. math. and math. phys.,1969,V.9,N 4, p.841-859.

[2] Liseikin V.D., Petrenko V.E. The Adaptive invariant method of the numerical solution of problems with boundary and interior layers. Novosibirsk ,1989.

[3] Vulanovic R.,Herceg D.,Petrovic N. On the extrapolation for a singularly perturbed boundary value problem.Computing ,1986, V.36,N 1-2,p.69-79.

[4] Herceg D. On numerical solution of singularly perturbed boundary value problem.Review of Research faculty of science mathematics series, Novi Sad,1993,V.23,N 1,p.371-381.

[5] Samarsky A.A. Theory of difference schemes. Moskow, 1977.

[6] Gaevoy V.P. About one method of difference scheme construction for two-point boundary value problems // Computing systems, Novosibirsk, 1978, N 75, p.96-110.

[7] Doolan E.P.,Miller J.J.H.,Schilders W.H.A. Uniform numerical methods for problems with initial and boundary layers.Dublin: Boole Press,1980.

[8] Zadorin A.I. The difference scheme for self-adjoint singularly perturbed third baundary value problem // Modelling in mechanics, Novosibirsk, 1989, V.3,N 1, p. 77-82.

[9] Kadalbajoo M.K., Reddy Y.N. Asymptotic and numerical analysis of singular perturbation problems: a survey// Appl. Math. and Comput.,1989, V.30, N 3, p. 223-259.

[10] Zadorin A.I.,Ignat'ev V.N. Numerical solution of a singularly perturbed second order quasilinear equation//Jornal of comput. math. and math. phys., 1991,V.31,N 1,p. 157-161.

[11] Ortega J.M., Rheinboldt W.C. Iterative solution of nonlinear equations in several variables, New York and London : Academic Press,1970.