NUMERICAL SOLUTION OF THE NONLINEAR DIFFERENTIAL EQUATION WITH A SMALL PARAMETER ON THE INFINITE INTERVAL

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Some physical processes, as flame spreading or pollution transfer lead to elliptic problems for unlimited domains. There is a problem to transfer the boundary conditions to the boundary of the limited domain. In this work we consider the case of half - limited interval. We use approach [1],[2] to extract stable set of solutions, corresponding to condition at the infinity. We suggest an asimptotic method to form the boundary conditions.

Define for any continuous or mesh function $p(x)$ $\quad ||p|| = \max|p(x)|$. Throughout the paper let C and C_i denote positive constants, that are always independent of ε and h.

Consider the problem:

$$T_\varepsilon u = -\varepsilon u'' + mu' + g(u) = 0, \tag{1}$$

$$u(0) = A, \lim_{x\to\infty} u(x) = B. \tag{2}$$

We suppose, that $g''(s)$ is continuous function,

$$\varepsilon \in (0,1],\ m > 0,\ g(B) = 0,\ g'(s) \geq \alpha > 0,\ s \in R. \tag{3}$$

Problem (1)-(2) simulates the flame spreading, when $u-$ the temperature, $m-$ flame speed, ε – diffusion coefficient, function $g(u)$ corresponds to Arrenius law:

$$g(u) = K(u - B)\exp(-E/u).$$

Get the estimate of stability for operator T_ε.

Lemma 1. Let $p(x)$, $q(x)$ - any enough smooth functions, bounded at the infinity. Then for all $x \in [0,\infty)$

$$|p(x) - q(x)| \leq \alpha^{-1}\|T_\varepsilon p - T_\varepsilon q\| + |p(0) - q(0)| + \lim_{s\to\infty}|p(s) - q(s)|.$$

Proof. Let $z = p - q$. Define linear operator:

$$L_\varepsilon z = -\varepsilon z'' + mz' + bz = T_\varepsilon p - T_\varepsilon q,\ b = [g(p) - g(q)][p - q]^{-1}.$$

Let

$$\Psi(x) = \alpha^{-1}\|T_\varepsilon p - T_\varepsilon q\| + |z(0)| + \lim_{s\to\infty}|z(s)| \pm z(x).$$

Then

$$L_\varepsilon \Psi(x) \geq 0,\ 0 < x < \infty,\ \Psi(0) \geq 0,\ \lim_{x\to\infty} \Psi(x) \geq 0.$$

Using maximum principle we get $\Psi(x) \geq 0$. Lemma is proved.

Taking $p = u$, $q = 0$, we have from lemma 1:

$$|u(x)| \leq \beta^{-1}|g(0)| + |A| + |B|.$$

Lemma 2. Function $u(x)$ increases, if $A < B$ and decreases, if $A > B$.

Proof. Let $A < B,\ z = u - B$. Define linear operator:

$$L_\varepsilon \phi = -\varepsilon\phi'' + m\phi' + [g(z+B) - g(B)]z^{-1}\phi. \tag{4}$$

If

$$\phi(0) \leq 0,\ \lim_{x\to\infty} \phi(x) = 0,\ L_\varepsilon\phi(x) = 0$$

, then $\phi(x) \leq 0$. Using $\phi(x) = z(x)$, we get $u(x) \leq B$. Using the relation:

$$u'(x) = -\varepsilon^{-1} \int_x^\infty [g(u(s)) - g(B)] \exp[m\varepsilon^{-1}(x-s)]ds,$$

we get $u'(x) \geq 0$ for any x. The case $A \geq B$ is the same. Lemma is proved.

Lemma 3. For any x

$$|u(x) - B| \leq |A - B| \exp[r_0 x],$$

where r_0 is the negative root of equation: $-\varepsilon r^2 + mr + \alpha = 0$.

We can proof lemma 3, using maximum principle for operator L_ε.

Transfer the boundary condition from the infinity to some point L_0. The point $(B, 0)$ is a special point of the type "sadle" in the space of the variables (u, u'). We use approach [1],[2] and define the stable separatrice of the "sadle" by relation:

$$u'(x) = r_1(u(x) - B) + \gamma(u(x)), \tag{5}$$

r_1 is the negative root of the equation $-\varepsilon r^2 + mr + g'(B) = 0$,

$\gamma(u)$ is solution of the problem:

$$\varepsilon\gamma'(u)[r_1(u - B) + \gamma(u)] = \varepsilon r_2\gamma(u) + g(u) - g'(B)(u - B),$$

$$\gamma(B) = 0, \quad r_1 + r_2 = m\varepsilon^{-1}. \tag{6}$$

Lemma 4. For all x:

$$|\gamma(u(x))| \leq C \exp\{2r_0 x\}. \tag{7}$$

Proof. Taking into account (5),(6), we get:

$$\varepsilon \frac{d}{dx}\gamma(u(x)) - r_2\varepsilon\gamma(u(x)) = g(u(x)) - g'(B)(u(x)-B).$$

It follows:

$$\gamma(u(x)) = -\varepsilon^{-1}\int\limits_x^{\infty}[g(u(s)) - g'(B)(u(s)-B)]\exp[r_2(x-s)]ds.$$

hence,

$$|\gamma(u(x))| \leq \max|g''_u(u(x))|(2r_2\varepsilon)^{-1}(u(x)-B)^2.$$

Using Lemma 3, we get (7).

Using the equation (5) we form the problem for any finite interval :

$$-\varepsilon u'' + mu' + g(u) = 0,$$

$$u(0) = A,\ u'(L_0) = r_1[u(L_0) - B] + \gamma(u(L_0)). \tag{8}$$

We can find $\gamma(u)$ from the equation (6) with some mistake. Investigate the influence of that mistake on the solution of problem (8).

heorem 1. Let $\tilde{u}$ is solution of problem (8) in the case of function $\tilde{\gamma}(v)$. Let $\tilde{\gamma}'(v)$ is continuous function, $\quad r_1 + \tilde{\gamma}'(v) \leq 0, \quad v \in R$. Let

$$|\gamma(u(L_0)) - \tilde{\gamma}(u(L_0))| \leq \Delta,$$

where $u(x)$ is solution of the problem (1)-(2). Then for every $x \in [0, L_0]$

$$|u(x) - \tilde{u}(x)| \leq 2\Delta\varepsilon m^{-1}\exp[m(2\varepsilon)^{-1}(x - L_0)]. \tag{9}$$

Proof. Let $z = u - \tilde{u}$. Then

$$L_\varepsilon z = -\varepsilon z'' + mz' + [g(u) - g(\tilde{u})](u-\tilde{u})^{-1}z = 0,$$

$$z(0) = 0,\ D_\varepsilon z = z'(L_0) + \tau z(L_0) = \gamma(u(L_0)) - \tilde{\gamma}(u(L_0)),$$

where $\tau = -r_1 - \tilde{\gamma}'(\theta)$ for some θ.

Define

$$\Psi(x) = 2\Delta\varepsilon m^{-1}\exp[m(2\varepsilon)^{-1}(x - L_0)] \pm z(x).$$

Then

$$L_\varepsilon\Psi(x) \geq 0,\ x \in (0, L_0), \quad \Psi(0) \geq 0,\ \Psi'(L_0) - (r_1 + \gamma_0'(\theta))\Psi(L_0) \geq 0.$$

Using maximum principle we get $\Psi(x) \geq 0, \quad x \in [0, L_0]$. The theorem is proved.

Let $\tilde{u}'(L_0) = 0$. Then

$$|u(x) - \tilde{u}(x)| \leq C\varepsilon \exp[m(2\varepsilon)^{-1}(x - L_0) + r_0 L_0], \ 0 \leq x \leq L_0.$$

Let $\tilde{u}(L_0) = B$. In this case

$$|u(x) - \tilde{u}(x)| \leq C \exp[m(2\varepsilon)^{-1}(x - L_0) + r_0 L_0], \ 0 \leq x \leq L_0.$$

We use asymptotic approach to solve problem (6). We consider two terms of asymptotic series:

$$\tilde{\gamma}(u) = \gamma_0(u) + \varepsilon\gamma_1(u). \tag{10}$$

Taking into account (10) in (6), we have:

$$\gamma_0(u) = \frac{g'(B)(u - B) - g(u)}{m},$$

$$\gamma_1(u) = -\frac{g'(B)}{m^2}\{\gamma_0(u) + g(u)\gamma_0'(u)\}. \tag{11}$$

Lemma 5. Let $g''(s)$ is continuous function, $u(x)$ is solution of problem (1)-(2). Then for all x

$$|\gamma(u(x)) - \tilde{\gamma}(u(x))| \leq C\varepsilon^2. \tag{12}$$

Proof. Let $z(u(x)) = \gamma(u(x)) - \tilde{\gamma}(u(x))$. Then

$$\varepsilon\frac{d}{dx}z(u(x)) - \varepsilon r_2 z(u(x)) = F(x), \quad \lim_{x\to\infty} z(u(x)) = 0, \tag{13}$$

where $|F(x)| \leq C_0\varepsilon^2$. From (13) we have

$$z(u(x)) = -\frac{1}{\varepsilon}\int_x^\infty F(s)\exp[r_2(x - s)]ds.$$

It follows that $|z(u(x))| \leq C_0 m^{-1}\varepsilon^2$. Lemma is proved.

The problem (8) is restriction of the problem (1)-(2) to a finite interval. We can prove, that for finite j for some C and all $x \in [0, L_0]$

$$|u^{(j)}(x)| \leq C. \tag{14}$$

So, the solution of the problem (8) has not boundary layer and some monotone scheme may be used to solve the problem (8). As we noted, the problem (6) can be solved with some mistake. Investigate the influence of that mistake on the solution of a one sided difference scheme.

Let Ω is the uniform mesh of the interval $[0, L_0]$. Consider the scheme:

$$T_n^h u^h = -\varepsilon \Lambda_{xx,n} u^h + a_n \Lambda_{x,n} u^h + g(u_n^h) = 0,$$

$$u_0^h = A, \ R_h u^h = \Lambda_{x,N} u^h + \Theta(u_N^h) = 0, \tag{15}$$

where

$$\Lambda_{x,n} u^h = \frac{u_n^h - u_{n-1}^h}{h}, \ \Lambda_{xx,n} u^h = \frac{u_{n+1}^h - 2u_n^h + u_{n-1}^h}{h^2}, \ \Theta(u) = -r_1(u - B) - \gamma(u).$$

Investigate the properties of scheme (15).

Lemma 6. Let p^h, q^h are any mesh functions . Then for every n:

$$|p_n^h - q_n^h| \le \alpha^{-1} |T^h p^h - T^h q^h| + |p_0^h - q_0^h| +$$

$$+ (4\varepsilon + 2\alpha h)\alpha^{-1} |R_h p^h - R_h q^h| \exp[\alpha(2\varepsilon + \alpha h)^{-1}(x_n - L_0)]. \tag{16}$$

Proof. Let $z^h = p^h - q^h$. Then

$$L^h z^h = T^h p^h - T^h q^h, \ z_0^h = p_0^h - q_0^h, \ D_h z^h = R_h p^h - R_h q^h, \tag{17}$$

where L^h – linear operator

$$L_n^h z^h = -\varepsilon \Lambda_{xx,n} z^h + a_n \Lambda_{x,n} z^h + b_n z_n^h, \tag{18}$$

$b_n = [f(p_n^h, x_n) - f(q_n^h, x_n)]/(p_n^h - q_n^h)$,

D_h – linear operator:

$$D_h z^h = [\Theta(p_N^h) - \Theta(q_N^h)][p_N^h - q_N^h]^{-1} z_N^h + \Lambda_{x,N} z^h.$$

Define mesh functions ϕ^h, ρ^h:

$$\phi_n^h = \left[1 + \frac{\alpha h}{2\varepsilon}\right]^{n-N}, \quad \rho_n^h = \left[1 + \frac{\alpha h}{2\varepsilon}\right]^{n+1-N}.$$

Then

$$\exp[\alpha(2\varepsilon)^{-1}(x_n - L_0)] \le \phi_n^h \le \exp[\alpha(2\varepsilon + \alpha h)^{-1}(x_n - L_0)].$$

$$L_n^h \phi^h \ge \sigma[8\varepsilon + 4\alpha h]^{-1} \phi_n^h, \quad D_h \phi^h \ge \alpha[4\varepsilon + 2\alpha h]^{-1}, \quad D_h \theta^h \ge 0. \tag{19}$$

Define Ψ^h:

$$\Psi^h = \alpha^{-1} \|T^h p^h - T^h q^h\| + |p_0^h - q_0^h| + (4\varepsilon + 2\alpha h)\alpha^{-1} |R_h p^h - R_h q^h| \phi^h \pm z^h.$$

Then

$$\Psi_0^h \ge 0, \ D_h N^h \Psi^h \ge 0, L_n^h \Psi^h \ge 0, \ n = 1, 2, ..., N-1, \tag{20}$$

Using maximum principle we get $\Psi^h \geq 0$. Lemma is proved. From lemma 6 follows that scheme (15) has unique and bounded solution.

According to the next theorem the scheme (15) has the property of the uniform convergence in ε.

Theorem 2. For some C

$$||[u]_\Omega - u^h|| \leq Ch. \tag{21}$$

We can prove that theorem, using (14) and maximum principle.

Lemma 7. Let $\tilde{\Theta}'(v)$ *is continuous function,* $\tilde{\Theta}'(V) \geq \tilde{\alpha} > 0$. *Let* $\tilde{u}^h$ *is solution of scheme (15) in the case of perturbed function* $\tilde{\Theta}$. *Then if*

$$|\Theta(u_N^h) - \tilde{\Theta}(u_N^h)| \leq \Delta,$$

then for every n

$$|u_n^h - \tilde{u}_n^h| \leq \Delta\eta^{-1}(4\varepsilon + 2\alpha h)\exp[\alpha(2\varepsilon + \alpha h)^{-1}(x_n - L_0)]. \tag{22}$$

Prove. Let $z^h = u^h - \tilde{u}^h$. Then

$$L_n^h z^h = 0,\ z_0^h = 0,\ D_h z^h = \frac{z_N^h - z_{N-1}^h}{h} + \frac{\tilde{\Theta}(u_N^h) - \tilde{\Theta}(\tilde{u}_N^h)}{u_N^h - \tilde{u}_N^h} z_N^h = \tilde{\Theta}(u_N^h) - \Theta(u_N^h),$$

L^h (3.7) $b_n = [f(u_n^h, x_n) - f(\tilde{u}_n^h, x_n)]/(u_n^h - \tilde{u}_n^h)$. Define Ψ^h:

$$\Psi_n^h = \Delta\eta^{-1}(4\varepsilon + 2\alpha h)\phi_n^h \pm z_n^h.$$

Using maximum principle we prove lemma.

Scheme (15) is a sistem of nonlinear equations. We use Picar method to find solution:

$$-\varepsilon\Lambda_{xx,n}u^{j+1} + a_n\Lambda_{x,n}u^{j+1} + Gu_n^{j+1} = Gu_n^j - f(u_n^j, x_n),$$
$$u_0^{j+1} = A,\ \Lambda_{x,N}u^{j+1} + Gu_N^{j+1} = Gu_N^j - \Theta(u_N^j). \tag{3.22}$$

Lemma 8. Let

$$0 < g \leq \frac{\partial f}{\partial u} \leq G, \quad 0 < g \leq \Theta'(u) \leq G.$$

Then for every iteration j

$$||u^{j+1} - u^h|| \leq \left(1 - \frac{g}{G}\right)||u^j - u^h||.$$

Proof. Define $z^j = u^j - u^h$. Then

$$-\varepsilon\Lambda_{xx,n}z^{j+1} + a_n\Lambda_{x,n}z^{j+1} + Gz_n^{j+1} = [G - f'_u(s_n, x_n)]z_n^j,$$

$$z_0^{j+1} = 0, \;\; \Lambda_{x,N}z^{j+1} + Gz_N^{j+1} = [G - \Theta'(r_n)]z_N^j.$$

Define

$$\Psi_n^h = \left(1 - \frac{g}{G}\right)||z^j|| \pm z_n^{j+1}.$$

Then the conditions (20) are fulfilled and therefore $\Psi^h \geq 0$. Lemma is proved.

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