Numerical Method for a Singular Perturbed Parabolic Equation in a Strip

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Abstract

Parabolic problems on a half-infinite interval and in a half-strip are considered. Method of lines is used to transform parabolic problems to a boundary value problem for a system of ordinary differential equations (ODE) on an half-infinite interval. Method of extraction of a set of solutions, satisfying the limit conditions at the infinity, is used to transform a problem on half-infinite interval to a problem for a finite interval. Asymptotic series are used to solve auxiliary Cauchy problems with conditions at an infinity.

1 Introduction

We consider a parabolic problems on unbounded domains - on a half-infinite interval and in a half-strip. For construction of a constructive for computer computations difference scheme we have to transform a problems under consideration to a problems for a bounded domains. Boundary value problems on unbounded domains were considered in works of A. Abramov and N. Konyukhova [1], A. Zadorin and O. Harina [2], L. Vulkov and M. Koleva [3], G. Shishkin [4] and in other works. We use approach of work [1], where was considered a system of ODE with a limit condition at infinity. At first we use method of lines to transform a parabolic problem to a problem for a system of second order ODE on half-infinite interval. Then we construct a set of solutions of the system, that satisfy the limit boundary condition at infinity. This set may be given by first order system of ODE. Last system we use as a boundary condition for a problem on bounded domain.

We understand, that C and C_i are positive constants, that don't depend on parameter ε.

2 A system of ODE on half-infinite interval

Consider a boundary value problem in a matrix form:

$$\varepsilon^2 \mathbf{U}''(x) - P(x)\mathbf{U}(x) = \mathbf{F}(x), \tag{1}$$

$$\mathbf{U}(0) = \mathbf{A}, \quad \lim_{x\to\infty} \mathbf{U}(x) = \mathbf{0}, \tag{2}$$

where $\mathbf{F}(x)$, $P(x)$ are smooth enough, $P(x)$ is positive definite matrix of order M,

$$P(x) \geq \alpha I, \;\; \alpha > 0 \;\; \varepsilon > 0.$$

According to [2] next estimate

$$\max_x ||\mathbf{U}(x)|| \leq \sqrt{\frac{1}{\alpha^2} \max_x ||\mathbf{F}(x)||^2 + ||\mathbf{A}||^2}, \;\; x \geq 0$$

is valid, where

$$||\mathbf{U}|| = \Big\{ \sum_{k=1}^{n} U_k^2 \Big\}^{1/2}.$$

We investigate method of reduction of a problem (1)-(2) to a problem for a finite interval, using method of extraction of a set of all solutions, satisfying the limit condition at infinity [1],[2].

Define first order system:

$$\varepsilon \mathbf{W}'(x) + G(x)\mathbf{W}(x) = \mathbf{m}(x), \tag{3}$$

where matrix $G(x)$ is solution of a singular Cauchy problem for the matrix Riccati equation:

$$\varepsilon G'(x) - G^2(x) + P(x) = 0, \; \lim_{x\to\infty} G(x) = \sqrt{P_\infty}, \tag{4}$$

vector-function $\mathbf{m}(x)$ is the solution of singular problem:

$$\varepsilon \mathbf{m}'(x) - G(x)\mathbf{m}(x) = \mathbf{F}(x), \; \lim_{x\to\infty} \mathbf{m}(x) = \mathbf{0}. \tag{5}$$

Problems (4), (5) are composed so, that every solution of equation (3) satisfies to equation (1). According to [2] for any $x \geq 0$

$$G(x) \geq \sqrt{\alpha} I.$$

Using maximum principle we can prove inequality

$$\max_{x\geq s} ||\mathbf{m}(x)|| \leq \frac{1}{\sqrt{\alpha}} \max_{x\geq s} ||\mathbf{F}(x)||, \quad s \geq 0$$

and next lemma.

Lemma 1 *The estimate*

$$\max_x ||\mathbf{W}(x)|| \leq \sqrt{\frac{1}{\alpha^2} \max_x ||\mathbf{m}(x)||^2 + ||\mathbf{W}(0)||^2}.$$

is valid.

Now we prove, that equation (3) extracts solutions of equation (1), satisfying the limit condition at infinity.

Lemma 2 *Let* $\mathbf{W}(x)$ *is solution of (3),*

$$G(x) \geq \beta I,\ \beta > 0,\ \mathbf{m}(x) \to \mathbf{0},\ x \to \infty.$$

Then

$$\mathbf{W}(x) \to \mathbf{0},\ x \to \infty.$$

Proof. Consider a problem:

$$\varepsilon \mathbf{V}'(x) + \beta \mathbf{V}(x) = \mathbf{m}(x),\ \mathbf{V}(0) = \mathbf{A}.$$

Integrating last equation, we have:

$$V_i(x) = A_i e^{-\beta\varepsilon^{-1}x} + \frac{1}{\varepsilon}\int\limits_0^x e^{-\beta\varepsilon^{-1}(s-x)} m_i(s)d\,s.$$

Taking into account, that $m_i(s) \to 0,\ s \to \infty$, we get:

$$V_i(x) \ \to 0,\ x \to \infty.$$

So, $\mathbf{V}(x) \ \to \mathbf{0},\ x \to \infty$. If we denote $\mathbf{Z}(x) = \mathbf{V}(x) - \mathbf{W}(x)$, then $\mathbf{Z}(x)$ is a solution of a problem:

$$\varepsilon \mathbf{Z}'(x) + \beta \mathbf{Z}(x) = [G(x) - \beta I]\mathbf{W}(x),\ \mathbf{Z}(0) = \mathbf{0}.$$

Let $w = (\mathbf{Z}, \mathbf{Z})$. Multiplying last equation on $\mathbf{Z}(x)$, we get:

$$\frac{\varepsilon}{2}w' + \beta w = ((G - \beta I)\mathbf{W}, \mathbf{V}) - ((G - \beta I)\mathbf{W}, \mathbf{W}) \leq ||(G - \beta I)\mathbf{W}||\,||\mathbf{V}||.$$

Let

$$\Upsilon(x) = ||(G - \beta I)\mathbf{W}(x)||\,||\mathbf{V}(x)||.$$

Then we have an inequality:

$$\frac{\varepsilon}{2}w'(x) + \beta w(x) \leq \Upsilon(x). \tag{6}$$

Taking into account, that $\mathbf{V}(x) \to \mathbf{0},\ \ x \to \infty$, we get:

$$\Upsilon(x) \to \mathbf{0},\ \ x \to \infty.$$

Integrating (6) from 0 to x, we get:

$$w(x) \leq \frac{2}{\varepsilon}\int\limits_0^x \Upsilon(s) e^{2\beta\varepsilon^{-1}(s-x)}\,d\,s.$$

It follows, that $w(x) \to 0,\ \ x \to \infty$, therefore $\mathbf{Z}(x) \ \to \mathbf{0},\ $ if $x \to \infty$. Lemma is proved.

Conditions of Lemma 2 are fulfilled for $\beta = \sqrt{\alpha}$, therefore, all solutions of (3) tend to zero at infinity. We proved, that equation (3) extracts solutions of equation (1), satisfying the limit boundary condition (2) at infinity. Using equation (3), reduce problem (1)-(2) to a problem for a finite interval:

$$\varepsilon^2\mathbf{U}''(x) - P(x)\mathbf{U}(x) = \mathbf{F}(x),\ 0 < x < L,$$
$$\mathbf{U}(0) = \mathbf{A},\ \varepsilon\mathbf{U}'(L) + G(L)\mathbf{U}(L) = \mathbf{m}(L). \tag{7}$$

We can prove, that problems (1)-(2) and (7) have a same solution $\mathbf{U}(x)$ for $0 \le x \le L$. We must solve a question, how to find $G(L)$ and $\mathbf{m}(L)$ from singular Cauchy problems (4), (5). We use asymptotic series on a small parameter ε :

$$G^j(x) = \sum_{k=0}^{j} G_k(x)\varepsilon^k, \quad \mathbf{m}^j(x) = \sum_{k=0}^{j} \mathbf{m}_k(x)\varepsilon^k. \tag{8}$$

Substituting expressions for $G^j(x)$ and $\mathbf{m}^j(x)$ into equations (4) and (5), respectively, we get recurrence formulas:

$$G_0(x)G_k(x) + G_k(x)G_0(x) = G'_{k-1}(x) - \sum_{i=1}^{k-1} G_i(x)G_{k-i}(x),\ k \ge 1,$$
$$G_0(x) = \sqrt{P(x)};$$
$$G_0(x)\mathbf{m}_k(x) = \mathbf{m}'_{k-1}(x) - \sum_{i=1}^{k} G_i(x)\mathbf{m}_{k-i}(x),\ k \ge 1,$$
$$G_0(x)\mathbf{m}_0(x) = -F(x).$$

Using that solutions of problems (4),(5) are stable to small perturbations of $P(x)$ and $F(x)$, we get estimates of accuracy:

$$\max_x ||G^j(x) - G(x)|| \le C\varepsilon^{j+1},\ \max_x ||\mathbf{m}^j(x) - \mathbf{m}(x)|| \le C\varepsilon^{j+1}.$$

On the other hand, for large enough $x \ge L$ we can seek solutions of problems (4), (5) in a form:

$$G^j(x) = \sum_{k=0}^{j} \frac{G_k}{x^k},\quad \mathbf{m}^j(x) = \sum_{k=0}^{j} \frac{\mathbf{m}_k}{x^k}.$$

And in this case we can get recurrence formulas on G_k, $\mathbf{m}_k$ and estimates of accuracy:

$$||G^j(x) - G(x)|| \le CL^{-(j+1)},\quad ||\mathbf{m}^j(x) - \mathbf{m}(x)|| \le CL^{-(j+1)},\quad x \ge L.$$

Using asymptotic series, we can find $G(L)$ and $\mathbf{m}(L)$ with given accuracy. According to the next theorem, solution of a problem (7) is stable to perturbations of $G(L)$ and $\mathbf{m}(L)$. It follows, that if we solve problems (4), (5) with some errors, it does not increase errors in solution of problem (7).

Theorem 1 *Let $\tilde{\mathbf{U}}(x)$ be solution of a problem (7) in the case of $\tilde{G}(L),\ \tilde{\mathbf{m}}(L)$. Let*

$$||G(L) - \tilde{G}(L)|| \le \Delta,\ ||\mathbf{m}(L) - \tilde{\mathbf{m}}(L)|| \le \Delta.$$

Then

$$\max_x ||\mathbf{U}(x) - \tilde{\mathbf{U}}(x)|| \le C\Delta \exp\left[\sqrt{\alpha}(2\varepsilon)^{-1}(x-L)\right].$$

3 Parabolic equation on infinite interval

Consider a problem:

$$-\frac{\partial u}{\partial t} + \varepsilon^2 \frac{\partial^2 u}{\partial x^2} - b(x,t)u = f(x,t), \tag{9}$$

$$u(0,t) = \phi_1(t), u(x,0) = \phi_2(x), \lim_{x\to+\infty} u(x,t) = 0. \tag{10}$$

Let solution $u(x,t)$ is enough smooth function,

$$\varepsilon > 0,\ b(x,t) \geq \gamma > 0,\ \lim_{x\to+\infty} \phi_2(x) = 0,\ \lim_{x\to+\infty} f(x,t) = 0,\quad 0 \leq t \leq T.$$

We use method of lines on variable t and transform (9)-(10) to a problem for a second order system of ODE:

$$-\frac{u^i(x) - u^{i-1}(x)}{\tau} + \varepsilon^2 \frac{d^2\, u^i}{d\,x^2} - b(x)u^i = f(x,t_i)$$

$$u^0(x) = \phi_2(x),\quad u^i(0) = \phi_1(t_i),\ \lim_{x\to+\infty} u^i(x) = 0,$$

where $t_i = \tau i, i = 1,2,\ldots,M, t_M = T$.

Taking into account, that derivative $|u''_{tt}(x,t)|$ is uniformly bounded, we can estimate the error of approximation:

$$\max_{i,x} |u(x,t_i) - u^i(x)| \leq C\tau.$$

Problem (9)-(10) for parabolic equation is transformed to the system of type (1)-(2) with the matrix

$$P(x) = \begin{pmatrix} b(x,t_1) + \frac{1}{\tau} & 0 & \ldots & 0 \\ -\frac{1}{\tau} & b(x,t_2) + \frac{1}{\tau} & \ldots & 0 \\ & \ldots & & \\ 0 & \ldots & -\frac{1}{\tau} & b(x,t_M) + \frac{1}{\tau} \end{pmatrix}.$$

Since matrix $P(x)$ is of low triangle form, we can write problem (7) for every i in scalar form:

$$-\frac{u^i(x) - u^{i-1}(x)}{\tau} + \varepsilon^2 \frac{d^2\, u^i}{d\,x^2} - b(x)u^i = f(x,t_i),\quad u^0(x) = \phi_2(x),\ 1 \leq i \leq M,$$

$$u^i(0) = \phi_1(t_i),\ \varepsilon \frac{d}{d\,x} u^i(L) + G_{i,i}u^i(L) = m_i(L) - \sum_{j=1}^{i-1} G_{i,j}(L)u^j(L). \tag{11}$$

Note, that in case of parabolic problem we can write explicit formulas for a calculation of $G(L)$ and $\mathbf{m}(L)$. Consider a case, when we construct $G(x)$, using asymptotic series (8). On each iteration for G_k we have to solve matrix equation of a form:

$$AX + XB = F. \tag{12}$$

Matrices A, B, X, F are of low triangle form and for every diagonal with elements $X_{i,i-p}$, $(p = 0,1,2,\ldots,i-1)$ we can find:

$$X_{i,i-p} = \left[F_{i,i-p} - \sum_{k=i-p}^{i-1} A_{i,k}X_{k,i-p} - \sum_{k=i+1-p}^{i} X_{i,k}B_{k,i-p} \right] / \big[A_{i,i} + B_{i-p,i-p}\big].$$

If P is low triangle matrix, we can use explicit formula to solve matrix equation $X^2 = P$, that we have to solve to get initial iteration $G_0(x)$:

$$X_{i,i-p} = \left[P_{i,i-p} - \sum_{k=i+1-p}^{i-1} X_{i,k} X_{k,i-p} \right] \left[X_{i,i} + X_{i-p,i-p} \right]^{-1},$$

$$p = 1, 2, \dots, i-1, \quad x_{i,i} = \sqrt{P_{i,i}}, \quad 1 \le i \le M.$$

4 Parabolic equation in a strip

Consider a problem:

$$-\frac{\partial u}{\partial t} + \varepsilon^2 \frac{\partial^2 u}{\partial x^2} + \varepsilon^2 \frac{\partial^2 u}{\partial y^2} - b(x,y)u = f(x,y,t), \tag{13}$$

$$u(x,0,t) = \phi_1(x,t), u(x,1,t) = \phi_2(x,t), \; u(x,y,0) = \phi_3(x,y),$$
$$u(0,y,t) = \phi_4(y,t), \; \lim_{x\to+\infty} u(x,y,t) = 0 \tag{14}$$

for a half-strip $D = \{0 \le x < \infty, \; 0 \le y \le 1\}$ and $0 \le t \le 1$. Suppose, that $\varepsilon > 0, \; b(x,y) \ge b_0 > 0$,

$$\lim_{x\to\infty} \phi_i(x,t) = 0, \; i = 1,2, \quad \lim_{x\to\infty} f(x,y,t) = 0, \; \lim_{x\to\infty} b(x,y) = b_+(y).$$

Solution of problem (13)-(14) has boundary layers at the boundaries $x = 0$, $y = 0, y = 1$. Introduce uniform mesh Ω_t on t with a step τ and nonuniform mesh Ω_y on y with steps h_j. Using approximation of derivatives on t and y, we get a system of differential-difference equations:

$$-\frac{V_j^i - V_j^{i-1}}{\tau} + \varepsilon^2 \frac{d^2}{dx^2} V_j^i + \varepsilon^2 \Lambda_{yy,j} V^i - b(x,y_j) V_j^i = f(x,y_j,t_i),$$

$$0 < j < N, \; 0 < i \le M, \quad V_j^i(0) = \phi_4(y_j,t_i), \; \lim_{x\to+\infty} V_j^i(x) = 0,$$

$$V_0^i(x) = \phi_1(x,t_i), \; V_N^i(x) = \phi_2(x,t_i), \; V_j^0(x) = \phi_3(x,y_j), \tag{15}$$

where

$$\Lambda_{yy,j} V^i = 2 \frac{h_j (V_{j+1}^i - V_j^i) - h_{j+1}(V_j^i - V_{j-1}^i)}{h_j h_{j+1}(h_j + h_{j+1})}.$$

To take into account boundary layers on y, we shall use nonuniform mesh from [5] or [6]. In case of mesh [5] we have estimate of accuracy

$$\max_{i,j,x} |U_j^i(x) - V_j^i(x)| \le \frac{C}{N^2} + C\tau,$$

where $\mathbf{U}(x) = [u(x,y,t)]_{\Omega_y \times \Omega_t}$. For every i problem(15) can be written in the form (1)-(2) with a matrix

$$P(x) = \begin{pmatrix} p_1 + \frac{2\varepsilon^2}{h_1 h_2} & -\frac{2\varepsilon^2}{h_1(h_1+h_2)} & \dots & 0 \\ -\frac{2\varepsilon^2}{h_2(h_2+h_3)} & p_2 + \frac{2\varepsilon^2}{h_2 h_3} & -\frac{2\varepsilon^2}{h_3(h_2+h_3)} & \dots \\ & \dots & & \\ 0 & \dots & \dots & p_M + \frac{2\varepsilon^2}{h_{M-1} h_M} \end{pmatrix},$$

where $p_i = b(x, y_i) + \tau^{-1}$. Matrix $P(x)$ is diagonal dominated M-matrix. It is known, that in this case $\sqrt{P(x)}$ is diagonal dominated M-matrix too. Iterative method to resolve matrix equation $X^2 = P$, when P is diagonal dominated matrix, is discussed in [2]. Using asymptotic series for $G(x)$, we can prove, that for enough large x or for enough small ε matrix $G(x)$ is diagonal dominated matrix. In case of three-diagonal matrix $P(x)$ we have not explicit formulas for calculation of $G(x)$ and $\mathbf{m}(x)$. Numerical methods for matrix equation (12) are investigated in [7].

5 Summary

We proposed numerical method for parabolic problems in unbounded domains. Preliminary we considered a system of second order ODE on half-infinite interval. To reduce this problem to a problem for a finite interval, we investigated A. Abramov approach and got some new estimates. For parabolic equation on half-infinite interval we applied method of lines to reduce a problem under consideration to a system of ODE, then we used investigated approach and got some explicit formulas to compose a boundary value problem on a finite interval. Then we used the same approach for a parabolic problem in a strip.

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