

# Numerical Method for Blasius Equation on an infinite Interval \*

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## 1 Introduction

Blasius problem on a half-infinite interval is considered. This problem has a place under mathematical modelling of viscid flow before thin plate. Blasius problem is a boundary value problem for a nonlinear third order ordinary differential equation on a half-infinite interval.

This problem was investigated in many articles. For example, G.I. Shishkin [1] studied asymptotic behavior of differential and difference solutions to get difference scheme with a finite number of nodes for enough long interval. We apply the method of marking of set of solutions, that satisfy the limit boundary condition at infinity to transform a problem under consideration to a problem for a finite interval [2]. To use that method we have problems, connected with nonlinearity of differential equation and with unbounded coefficient before second derivative. We offer to do transformation of independent variable to avoid a problem, connected with boundlessness of that coefficient.

We consider a Blasius problem as model nonlinear problem for application of developing technic.

## 2 Case of a linear problem

Consider a problem:

$$\varepsilon u''(x) + [a(x) + x]u'(x) = f(x), \quad (1)$$

$$u(0) = A, \quad \lim_{x \rightarrow \infty} u(x) = 0. \quad (2)$$

Suppose, that  $a(x) + x \geq 0$ ,  $\varepsilon > 0$ , functions  $a(x)$ ,  $f(x)$  are smooth enough,

$$\exists \lim_{x \rightarrow \infty} a(x) = a_0, \quad \lim_{x \rightarrow \infty} f(x) = 0.$$

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Note, that domain is unbounded and solution  $u(x)$  can be unbounded for a bounded function  $f(x)$ . Integrating equation (1), we get, that if  $a(x) + x \geq \alpha > 0$ , then

$$|u(x)| \leq |A| + \frac{1}{\alpha} \int_0^x |f(s)| ds.$$

To avoid unboundedness of coefficient before first derivative, introduce new variable  $t = \frac{x^2}{2}$ . Then problem (1)-(2) become a form:

$$\begin{aligned} \varepsilon u''(t) + b(t)u'(t) &= F(t), \\ u(0) = A, \quad \lim_{t \rightarrow \infty} u(t) &= 0, \end{aligned} \quad (3)$$

where

$$b(t) = \frac{a(\sqrt{2t})}{\sqrt{2t}} + 1 + \frac{1}{2t}, \quad F(t) = f(\sqrt{2t})/(2t).$$

So, coefficient  $b(t)$  is bounded for  $t$ , separated from zero. We can do some other replacement of  $x$  to do function  $b(t)$  uniformly bounded, but it is not important for method, considered below.

By the next equation we mark solutions of differential equation (3), that satisfy the limit boundary condition at infinity [2, 3, 6]:

$$\varepsilon u'(t) + g(t)u(t) = \beta(t), \quad (4)$$

where  $g(t)$  is solution of singular Cauchy problem for Riccati equation:

$$\varepsilon g'(t) + b(t)g(t) - g^2(t) = 0, \quad \lim_{t \rightarrow \infty} g(t) = 1, \quad (5)$$

$\beta(t)$  is solution of singular problem:

$$\varepsilon \beta'(t) + [b(t) - g(t)]\beta(t) = \varepsilon F(t), \quad \lim_{t \rightarrow \infty} \beta(t) = 0. \quad (6)$$

Thank to condition  $\lim_{t \rightarrow \infty} g(t) = 1$  every solution of equation (4) tends to zero at infinity and equation (4) picks that solutions of differential equation (3), that satisfy the limit boundary condition at infinity.

**Lemma 1** *Let  $0 < b_1 \leq b(t) \leq b_2$ . Then  $b_1 \leq g(t) \leq b_2$ .*

**Proof.** At first prove, that  $g(t) > 0$ ,  $0 \leq t < \infty$ . Define

$$v(t) = \exp \left\{ - \int_0^t \varepsilon^{-1} g(s) ds \right\}.$$

Obviously,  $v(t)$  is solution of a problem:

$$\begin{aligned} \varepsilon v'' + b(t)v'(t) &= 0, \\ v(0) = 1, \quad \lim_{t \rightarrow \infty} v(t) &= 0, \end{aligned} \quad (7)$$

It follows from (7), that

$$v(t) > 0, v'(t) < 0, 0 < t < \infty.$$

Taking into account that  $g(t) = -\varepsilon v'(t)/v(t)$ , we get  $g(t) > 0, 0 \leq t < \infty$ .

Prove, that  $g(t) \leq b_2$ . Consider a problem:

$$\varepsilon g_2' + b_2 g_2 - g_2^2 = 0, \lim_{t \rightarrow \infty} g_2(t) = b_2$$

with solution  $g_2(t) = b_2$ . Let  $z = g_2 - g$ . Then  $z(t)$  is solution of a problem:

$$\varepsilon z'(t) - [g_2 - b(t) + g(t)]z(t) = (b(t) - g_2)g_2 \leq 0, \lim_{t \rightarrow \infty} z(t) = b_2 - b_\infty \geq 0. \quad (8)$$

If we suppose, that for some  $s$   $z(s) < 0$ , then we shall have contradiction. It follows, that  $g(t) \leq b_2$ . In the same manner we can prove, that  $g(t) \geq b_1$ . Lemma is proved.

Using a problem (6), we can prove, that

$$\|\beta\|_\infty \leq \|g\|_\infty \int_0^\infty |F(s)| ds.$$

Using equation (4), transform a problem (3) to a problem for a finite interval:

$$\begin{aligned} \varepsilon u''(t) + b(t)u'(t) &= f(t), \quad 0 < t < L, \\ u(0) = A, \quad \varepsilon u'(L) + g(L)u(L) &= \beta(L). \end{aligned} \quad (8)$$

Prove, that problems (3) and (8) have a same solution for  $0 \leq t \leq L$ . Consider initial value problem:

$$\varepsilon u'(t) + g(t)u(t) = \beta(L), \quad u(0) = A. \quad (9)$$

Taking into account equations (5),(6), we prove, that solution of a problem (9) satisfies to problems (3) and (8). Problems under consideration have unique solution, therefor problems (3) and (8) have a same solution for  $0 \leq t \leq L$ .

So, problem (3) is exactly reduced to a problem (8), formulated for a finite interval. We have only to find coefficients  $g(L)$  and  $\beta(L)$ .

Functions  $g(t)$ ,  $\beta(t)$  as solutions of problems (5),(6) we can sick as asymptotic series:

$$g(t) \approx \sum_{n=0}^N \varepsilon^n g_n(t), \quad \beta(t) \approx \sum_{n=0}^N \varepsilon^n \beta_n(t), \quad (10)$$

or in a form:

$$g(t) \approx \sum_{n=0}^N (2t)^{-n/2} g_n, \quad \beta(t) \approx \sum_{n=0}^N (2t)^{-n/2} \beta_n. \quad (11)$$

In case of representation (11) we suppose, that

$$\begin{aligned} a(\sqrt{2t}) &= \sum_{n=0}^N (2t)^{-n/2} a_n + O\left((2t)^{-(N+1)/2}\right), \\ f(\sqrt{2t}) &= \sum_{n=0}^N (2t)^{-n/2} f_n + O\left((2t)^{-(N+1)/2}\right). \end{aligned}$$

In both cases (10) and (11) we got recurrent formulas on  $\beta_n, g_n$ . In particular, in case of representation (11) we have:

$$g_0 = 1, \quad g_1 = a_0, \quad g_2 = 1 + a_1, \quad g_3 = -\varepsilon a_0 + 4a_0 + a_2 + 4a_0 a_1,$$

$$\beta_0 = 0, \quad \beta_1 = -f_1, \quad \beta_2 = -f_2 - \beta_1 \varepsilon^{-1} (g_3 - a_2).$$

Make inverse transformation of independent variable and get a next problem for a finite interval:

$$\varepsilon u''(x) + [a(x) + x]u'(x) = f(x), \quad 0 < x < L_0,$$

$$u(0) = A, \quad \frac{\varepsilon}{L_0} u'(L_0) + g(L)u(L_0) = \beta(L_0). \quad (12)$$

Using maximum principle, we can prove next lemma.

**Lemma 2** *Let  $\tilde{u}(t)$  is solution of problem (12) with perturbed coefficients  $\tilde{g}(L_0), \tilde{\beta}(L_0)$ ,*

$$|g(L_0) - \tilde{g}(L_0)|, \quad |\beta(L_0) - \tilde{\beta}(L_0)| \leq \Delta, \quad \tilde{g}(L_0) \geq \sigma > 0.$$

*Then*

$$|u(x) - \tilde{u}(x)| \leq \sigma^{-1} \Delta (|u(L_0)| + 1).$$

Consider a problem for numerical experiments:

$$u''(x) + \left[ \frac{2x}{x+1} + x \right] u'(x) = 0,$$

$$u(0) = 1, \quad \lim_{x \rightarrow \infty} u(x) = 0.$$

We compare different approaches for formulation of boundary condition at a finite point instead the limit boundary condition at infinity. Let  $\Delta = \max_n |v_n^h - \tilde{v}_n^h|$ , where  $\tilde{v}_n^h$  is solution of the scheme of upwind differences for enough long interval  $[0, L_0]$ ,  $L_0 = 100$ ;  $v_n^h$  is solution of the same difference scheme on a shot interval depending on boundary condition. In Table 1 error  $\Delta$  is represented for different approaches and intervals.

Table 1: Absolute errors for a linear problem

$L$	$w_n(L) = 0$	$\tilde{g}_n(L)$		
		$g_0$	$g_0 + g_1/L$	$g_0 + g_1/L + g_2/L^2$
2	$0.12e - 01$	$0.95e - 02$	$0.11e - 02$	$0.44e-04$
3	$0.35e - 03$	$0.20e - 03$	$0.19e - 04$	$0.41e-05$
5	$0.36e - 07$	$0.13e - 07$	$0.95e - 09$	$0.84e-10$
7	$0.32e - 12$	$0.86e - 13$	$0.49e - 14$	$0.23e-15$

### 3 Blasius problem

Consider Blasius problem:

$$\begin{aligned} u'''(x) + u(x)u''(x) &= 0, \\ u(0) = 0, \quad u'(0) &= 0, \quad \lim_{x \rightarrow \infty} u'(x) = 1. \end{aligned} \quad (13)$$

Let

$$u(x) = v(x) + x, \quad w(x) = v'(x).$$

Then problem (13) can be written in a form:

$$\begin{aligned} v'(x) &= w(x), \quad v(0) = 0, \\ w''(x) + [v(x) + x]w'(x) &= 0, \\ w(0) &= -1, \quad \lim_{x \rightarrow \infty} w(x) = 0. \end{aligned} \quad (14)$$

Consider iterative method for a problem (14):

$$\begin{aligned} v'_n(x) &= w_n(x), \quad v_n(0) = 0, \\ w''_n(x) + [v_{n-1}(x) + x]w'_n(x) &= 0, \quad 0 < x < \infty, \\ w_n(0) &= -1, \quad \lim_{x \rightarrow \infty} w_n(x) = 0. \end{aligned} \quad (15)$$

**Lemma 3** *Let*

$$v_0(x) \geq v(x), \quad v_0(x) \geq v_1(x), \quad x > 0.$$

*Then iterative method (15) converges.*

**Proof.** Prove, that for every value of  $x$  sequences  $v_n(x)$ ,  $w_n(x)$  are monotone decreasing and have low bounds.

*Low bounds.* Let  $z_n(x) = v_n(x) - v(x)$ ,  $p_n(x) = w_n(x) - w(x)$ . Compose a problem:

$$\begin{aligned} z'_n(x) &= p_n(x), \quad z_n(0) = 0, \\ p''_n(x) + [v_{n-1}(x) + x]p'_n(x) &= -z_{n-1}(x)w'(x), \quad 0 < x < \infty, \\ p_n(0) &= 0, \quad \lim_{x \rightarrow \infty} p_n(x) = 0. \end{aligned}$$

By induction prove, that for all  $x > 0$   $z_n(x) \geq 0$ ,  $p_n(x) \geq 0$ . According to conditions of lemma  $z_0(x) \geq 0$ . Let  $z_{n-1}(x) \geq 0$ . Prove, that  $p_n(x) \geq 0$ . Suppose, that for some  $s$   $p_n(s) < 0$ . Then there is point of negative minimum of function  $p_n(x)$ , it leads us to contradiction. So,  $p_n(x) \geq 0$ . It follows, that  $z_n(x) \geq 0$ . According to method of mathematical induction for all  $n$   $z_n(x) \geq 0$ ,  $p_n(x) \geq 0$ . It implies, that  $v_n(x) \geq v(x)$ ,  $w_n(x) \geq w(x)$ .

*Monotony.* Prove, that sequences  $v_n(x)$ ,  $w_n(x)$  are monotone decreasing.

Let  $z_n(x) = v_n(x) - v_{n+1}(x)$ ,  $p_n(x) = w_n(x) - w_{n+1}(x)$ . Then  $z_n(x)$   $p_n(x)$  are solutions of a problem:

$$\begin{aligned} z'_n(x) &= p_n(x), \quad z_n(0) = 0, \\ p''_n(x) + [v_{n-1}(x) + x]p'_n(x) &= -z_{n-1}(x)w'_{n+1}(x), \quad 0 < x < \infty, \\ p_n(0) &= 0, \quad \lim_{x \rightarrow \infty} p_n(x) = 0. \end{aligned}$$

Using method of induction, prove, that  $z_n(x) \geq 0$ ,  $p_n(x) \geq 0$ . According to conditions of lemma  $z_0(x) \geq 0$ . . It is follows from equation on  $p_n(x)$ , that if  $z_{n-1}(x) \geq 0$ , then  $p_n(x) \geq 0$  for  $x > 0$ . Condition  $p_n(x) \geq 0$  implies, that  $z_n(x) \geq 0$ . So, using method of mathematical induction, we proved, that for every  $n$   $z_n(x) \geq 0$ ,  $p_n(x) \geq 0$ .

We proved, that for every  $x > 0$  sequences  $v_n(x)$ ,  $w_n(x)$  are monotone decreasing and have low bounds. It's known, that in this case sequences under consideration have the property of convergence. Lemma is proved.

Consider a case  $v_0(x) = 0$ . Prove, that conditions of a lemma 3 are fulfilled. First condition has a place, because  $v(x) \leq 0$ . Verify the condition  $v_0(x) \geq v_1(x)$ ,  $x > 0$ . It follows from (15), that  $w_1(x) \leq 0$  for  $x > 0$ . It implies, that  $v_1(x) \leq 0 = v_0(x)$ . So, conditions of lemma 3 are valid.

Consider a question of reduction Blasius problem to a finite interval. For every fixed  $n$  problem (15) is linear and we can use results, obtained in linear case. Second equation in (15) corresponds to (1) with  $a(x) = v_{n-1}(x)$ .

Transformed to a finite interval problem (15) has a form:

$$\begin{aligned} v_n'(x) &= w_n(x), \quad v_n(0) = 0, \\ w_n''(x) + [v_{n-1}(x) + x]w_n'(x) &= 0, \quad 0 < x < L, \\ w_n(0) &= -1, \quad \frac{1}{L}w_n'(L) + g_n(L)w_n(L) = 0. \end{aligned} \quad (16)$$

Coefficient  $g_n(L)$  can be calculated on base of asymptotic series (11) as it was discussed for linear case.

Consider results of numerical experiments. Write difference scheme for a problem (16):

$$\begin{aligned} v_n^k &= v_n^{k-1} + \frac{w_n^{k-1} + w_n^k}{2}h, \quad v_n^0 = 0, \\ \frac{w_n^{k+1} - 2w_n^k + w_n^{k-1}}{h^2} + [v_{n-1}^k + x_k] \frac{w_n^{k+1} - w_n^k}{h} &= 0, \\ w_n^0 &= -1, \quad \frac{1}{L} \frac{w_n^K - w_n^{K-1}}{h} + \tilde{g}_n(L)w_n^K = 0, \\ k &= 1, 2, \dots, K - 1. \end{aligned} \quad (17)$$

Compare errors, corresponding to different approaches for formulation of boundary condition instead of limit boundary condition at infinity. Let

$$\Delta = \max_n |v_n^h - \tilde{v}_n^h|,$$

where  $\tilde{v}_n^h$  is solution of problem (17) for enough long interval  $[0, L_0]$ ,  $L_0 \gg L$ , when error of limit boundary condition transfer to point  $L_0$  is inessential. Let  $L_0 = 100$ ,  $h = 0.1$  We continue iterations, if  $\max_k |v_n^k - v_{n-1}^k| > \delta$ ,  $\delta = 10^{-13}$ . Initial iteration is defined as  $v_0(x) = 0$ . In Table 2 error  $\Delta$  is presented for different  $L$  and  $\tilde{g}_n(L)$  in compare with condition  $w_n(L) = 0$ . In this Table  $g_0 = 1$ ,  $g_1 = v_n(L)$ . We have similar results in case  $h = 0.01$ .

Table 2: Absolute errors for Blasius problem

$L$	$w_n(L) = 0$	$\tilde{g}_n(L) = g_0$	$\tilde{g}_n(L) = g_0 + g_1/L$
3	$0.83e - 1$	$0.24e - 1$	$0.19e - 1$
5	$0.76e - 3$	$0.15e - 3$	$0.47e - 4$
7	$0.34e - 6$	$0.53e - 7$	$0.97e - 8$
9	$0.12e - 10$	$0.15e - 11$	$0.18e - 12$

It follows from numerical experiments, that using special boundary condition, based on equation (4), we get more accurate results in compare with classical approach.

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